COMBINATORIAL RIGIDITY FOR SOME INFINITELY RENORMALIZABLE UNICRITICAL POLYNOMIALS

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Abstract. We prove combinatorial rigidity of *infinitely renormalizable* unicritical polynomials, $P_c : z \mapsto z^d + c$, with $c \in \mathbb{C}$, under the *a priori bounds* and a certain "combinatorial condition". This implies the local connectivity of the connectedness loci (the Mandelbrot set when $d = 2$) at the corresponding parameters.

1. Introduction

The *Multibrot set* $\mathcal{M}_d$ or the *connectedness locus* of the unicritical polynomials is the set of parameter values c in $\mathbb{C}$ for which the *Julia set* of $P_c : z \mapsto z^d + c$ is connected. $\mathcal{M}_2$ is the well-known *Mandelbrot* set.

There is a way of defining graded partitions of the Multibrot set into pieces such that dynamics of the maps P_c in each piece have some special combinatorial property. All maps in a given piece of a partition of a certain level are called *combinatorially equivalent* up to that level. Conjecturally, combinatorially equivalent (up to all levels) *non-hyperbolic* maps in this family are *conformally conjugate*. As stated in [DH84] for $d = 2$, this *rigidity conjecture* is equivalent to the local connectivity of the Mandelbrot set, and it naturally extends to degree d unicritical polynomials. In the quadratic case, this conjecture is formulated as MLC by A. Douady and J.H. Hubbard. They also proved there that MLC implies density of *hyperbolic polynomials* in the space of quadratic polynomials. These discussions have been extended to degree d unicritical polynomials by D. Schleicher in [Sch04].

In 1990's, J. C. Yoccoz proved MLC conjecture at all non-hyperbolic parameter values which are at most finitely *renormalizable*. He also proved local connectivity of the Julia sets of these maps with all periodic points repelling (see [Hub93]). Degree two assumption was essential in his proof.

2000 *Mathematics Subject Classification.* Primary 37F45, 37F25; Secondary 37F30.

In [Lyu97], M. Lyubich proved the combinatorial rigidity conjecture for a class of infinitely renormalizable quadratic polynomials. These are quadratic polynomials satisfying a *secondary limbs condition*, denoted by $\mathcal{SL}$, with sufficiently high return times. The proof in this case also relies on the degree two assumption.

Local connectivity of the Julia sets of degree d unicritical polynomials which are at most finitely renormalizable and with all periodic points repelling has been shown in [KL09]. Their proof is based on "controlling" geometry of a *modified principal nest*. The same controlling technique has been used to settle the rigidity problem for these parameters in [AKLS05].

Recently, the *a priori* bounds property, a type of compactness on renormalization levels, has been established for more parameters. In [Kah06] it is proved for infinitely *primitively* renormalizable maps of bounded type, in [KL08] it is proved for parameters under a *decorations* condition and in [KL07] under a *molecule* condition. Here we prove that the *a priori* bounds property, under $\mathcal{SL}$ condition, implies the combinatorial rigidity conjecture for infinitely renormalizable maps. The $\mathcal{SL}$ class includes all parameters for which the *a priori* bounds is known to us.

Theorem (Rigidity)**.** Let P_c be an infinitely renormalizable degree d unicritical polynomial satisfying the *a priori bounds* and $\mathcal{SL}$ conditions. Then P_c is combinatorially rigid.

This result was proved in part II of [Ly97] for quadratic polynomials. That proof as well as the one presented here are based on the Sullivan-Thurston pull-back argument. However, the one in [Ly97] uses linear growth of certain moduli along the principal nest which does not hold for arbitrary degree unicritical polynomials. It turns out that definite modulus of certain annuli in a modified principal nest introduced in [AKLS] helps us to "pass" over the principal nest much easier. This makes the whole construction simpler and more general to include arbitrary degree unicritical polynomials. Combining the above theorem with [KL08] and [KL07] we obtain the following:

Corollary. Assume that P and $\tilde{P}$ are combinatorially equivalent infinitely renormalizable unicritical polynomials that satisfy one of the following conditions:

– P and $\tilde{P}$ are quadratic and satisfy the molecule condition, or,

– P and $\tilde{P}$ have arbitrary degree and satisfy the decoration condition.

Then, P and $\tilde{P}$ are conformally equivalent.

The rigidity problem for a separate combinatorial class of quadratics is treated by a wholly different approach in [Lev09] which does not involve the *a priori* bounds property.

The rigidity problem for real polynomials is well developed. The quadratic case was accomplished, independently, in [Lyu97] and [GŚ98]. The real multi critical case was treated in [LvS98]. One may refer to these for further references. Our result applies to real unicritical polynomials as well. Therefore, combining with [Sul92], it gives a new proof of the density of hyperbolicity in the family $x \to x^{2k} + c$, $k = 1, 2, \dots$, which was proved earlier in [KSvS07].

The structure of the paper is as follows. In §2 we introduce the basics of holomorphic dynamics required for our work. In §3, Yoccoz puzzle pieces are defined, the modified principal nest is introduced, and combinatorics of unicritical polynomials is discussed. Proof of the main theorem, presented in Section 4, is reduced to existence of a *Thurston conjugacy* by the pull-back method. To build such a conjugacy, we start with a topological conjugacy on the whole complex plane and then step by step, on finer and finer scales, replace this homeomorphism by quasi-conformal maps while sacrificing equivariance property but staying in the "right" homotopy class. At the end one obtains a quasi conformal map on the complex plane homotopic to a topological conjugacy relative the post critical set, that is, a Thurston conjugacy.

Acknowledgment. I am indebted to M. Lyubich for suggesting the problem; this paper would have never been finished without his great patience. Further thanks is due to R. Pérez for our very useful discussions on the combinatorics of the Mandelbrot set.

2. Polynomials and the connectedness loci

2.1. **External rays and Equipotentials.** One can read more about the following basics of holomorphic dynamics in [Mil06] and [Bra94].

Let $f : \mathbb{C} \to \mathbb{C}$ be a monic polynomial of degree d, $f(z) = z^d + a_1 z^{d-1} + \dots + a_d$. Infinity is a super attracting fixed point of f whose *basin of attraction* is defined as

$$D_f(\infty) := \{z \in \mathbb{C} : f^n(z) \to \infty\}.$$

Its complement is called the *filled Julia* set: $K(f) = \mathbb{C} \setminus D_f(\infty)$. The *Julia set*, $J(f)$, is the common boundary of $K(f)$ and $D_f(\infty)$. It is well-known that the Julia set and the filled Julia set of a polynomial are connected if and only if all critical points stay bounded under iteration.

With f as above, there exists a conformal change of coordinate, *Böttcher coordinate*, B_f which conjugates f to the *dth* power map $z \mapsto z^d$ throughout some neighborhood of infinity U_f. That is,

$$B_f : U_f \to \{z \in \mathbb{C} : |z| > r_f \geq 1\} \tag{2.1}$$

with $B_f(f(z)) = (B_f(z))^d$, and $B_f(z) \sim z$ as $z \to \infty$.

In particular, if the filled Julia set is connected, B_f coincides with the Riemann mapping of $D_f(\infty)$ onto the complement of the closed unit disk normalized to be tangent to the identity map at infinity.

The *external ray* (or *ray* for short) *of angle* θ is defined as

$$R^\theta = R^\theta_f := B_f^{-1}\{re^{i\theta} : r_f < r < \infty\}.$$

The *equipotential of level* $r > r_f$ is defined as

$$E^r = E^r_f := B_f^{-1}\{re^{i\theta} : 0 \leq \theta \leq 2\pi\}.$$

Equivariance property of the map B_f implies that $f(R^\theta) = R^{d\theta}$, and $f(E^r) = E^{r^d}$.

A ray R^θ is called *periodic ray* of period p if $f^p(R^\theta) = R^\theta$. A ray is fixed (has period 1) if and only if θ is a rational number of the form $2\pi j/(d-1)$. By definition, a ray R^θ lands at a well defined point z in $J(f)$, if the limiting value of the ray R^θ (as $r \to r_f$) exists and is equal to z. Such a point z in $J(f)$ is called the *landing point* of the ray R^θ. The following theorem characterizes the landing points of the periodic rays. See [DH84, DH85a] for further discussions.

Theorem 2.1. *Let f be a polynomial of degree $d \geq 2$ with connected Julia set. Every periodic ray lands at a well defined periodic point which is either repelling or parabolic. Vice versa, every repelling or parabolic periodic point is the landing point of at least one, and at most finitely many periodic rays with the same ray period.*

In particular, this theorem implies that the external rays landing at a periodic point are organized in several cycles. Suppose $\overline{a} = \{a_k\}_{k=0}^{p-1}$ is a repelling or parabolic cycle of f. Let $\mathfrak{R}(a_k)$ denote the union of all the external rays landing at a_k. The configuration

$$\mathfrak{R}(\overline{a}) = \bigcup_{k=0}^{p-1}(\mathfrak{R}(a_k)),$$

with the rays labeled by their external angles, is called the *periodic point portrait* of f associated to the cycle $\overline{a}$.

2.2. **Unicritical family and the connectedness locus.** Any degree d polynomial with only one critical point is affinely conjugate to some $P_c(z) = z^d + c$, with $c \in \mathbb{C}$. A case of special interest is the following fixed point portrait. The $d-1$ fixed rays $R^{2\pi j/(d-1)}$ land at $d-1$ (distinct) fixed points called β_j. Moreover, these are the only rays that land at β_j's. These fixed points are *non-dividing* which means that $K(P_c) \setminus \beta_j$ is connected for any j. If the other fixed point called α is also repelling, there are at least 2 rays that land at this fixed point. Thus, the α-fixed point is *dividing* and by Theorem 2.1, these rays landing at α are permuted under the dynamics. The following statement has been shown in [Mil00b] for quadratic polynomials. The same ideas apply to prove it for degree d unicritical polynomials. See Figure 3 for the following proposition.

Proposition 2.2. *If at least* 2 *rays land at the* α *fixed point of* P_c, *we have:*

- *The component of* $\mathbb{C} \setminus P_c^{-1}(\mathfrak{R}(\alpha))$ *containing the critical value is a sector bounded by two external rays.*
- *The component of* $\mathbb{C} \setminus P_c^{-1}(\mathfrak{R}(\alpha))$ *containing the critical point is a region bounded by* $2d$ *external rays landing in pairs at the points* $e^{2\pi j/d}\alpha$, *for* $j = 0, 1, \dots, d-1$.

The *Connectedness locus* $\mathcal{M}_d$ is defined as the set of parameters c in $\mathbb{C}$ for which $J(P_c)$ is connected. In particular, $\mathcal{M}_2$ is the well-known *Mandelbrot* set. See Figures 1 and 2.

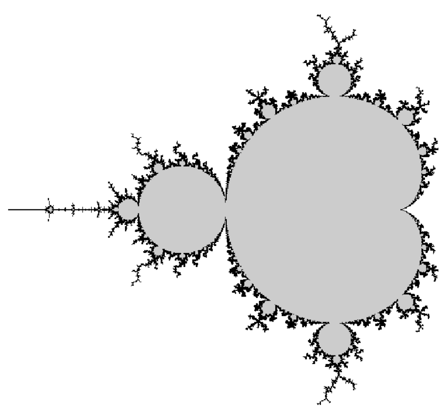

FIGURE 1. The Mandelbrot set. The gray regions show the interior of $\mathcal{M}_2$ and darker points show its boundary

A well-known result due to Douady and Hubbard [DH84] shows that these connectedness loci are connected. Their argument is based on

considering the explicit conformal isomorphism

$$\mathcal{B}_d : \mathbb{C} \setminus \mathcal{M}_d \to \{z \in \mathbb{C} : |z| > 1\}$$

given by $\mathcal{B}_d(c) = B_c(c)$, where B_c is the Böttcher coordinate (2.1) of P_c.

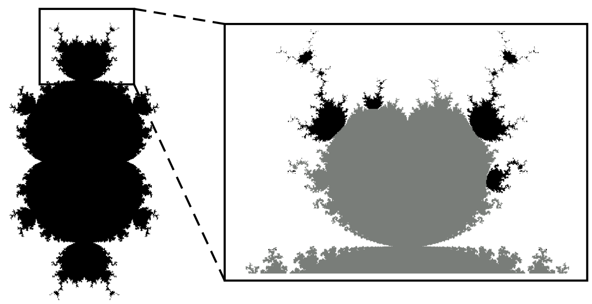

FIGURE 2. Figure on the left shows the connectedness locus $\mathcal{M}_3$. The figure on the right is an enlargement of a primary limb in $\mathcal{M}_3$. The dark regions in the right box show some of the secondary limbs

By means of the conformal isomorphism $\mathcal{B}_d$, parameter external rays R_θ and equipotentials E_r are defined, similarly, as $\mathcal{B}_d$-preimages of straight rays going to infinity and round circles around 0, respectively.

A polynomial P_c (also the corresponding parameter c) is called *hyperbolic* if P_c has an attracting periodic point. This attracting fixed point necessarily attracts orbit of the finite critical point. The set of hyperbolic parameters in $\mathcal{M}_d$, which is open by definition, is a union of some components of $\operatorname{int} \mathcal{M}_d$. These components are called *hyperbolic components.*

The main hyperbolic component is defined as the set of parameter values c for which P_c has an attracting fixed point. Outside of the closure of this set all the fixed points become repelling. Consider a parameter c in a hyperbolic component $\mathscr{H} \subset \operatorname{int} \mathcal{M}_d$, and suppose that $\overline{b}_c$ denotes the corresponding attracting cycle with period $k > 1$. On the boundary of $\mathscr{H}$ this cycle becomes neutral, and there are $d-1$ parameters $c_i \in \partial\mathscr{H}$ where P_{c_i} has a *parabolic* cycle with multiplier equal to one. One of these parameters, which we call it the *root* of $\mathscr{H}$ and denote it by c_{root}, divides the connectedness locus into two pieces. See [DH84] for quadratic polynomials and [Sch04] for arbitrary degree unicritical polynomials. Indeed, any hyperbolic component has one root and $d-2$ *co-roots.* The root is the landing point of two parameter

rays, while every co-root is the landing point of a single parameter ray. See Figure 3.

If c belongs to a hyperbolic component $\mathscr{H}$ that is not the main hyperbolic component of the connectedness locus, the *basin of attraction* of its attracting cycle $\bar{b}_c$, denoted by A_c, is defined as the set of points $z \in \mathbb{C}$ with $P_c^n(z)$ converges to the cycle $\bar{b}_c$. The boundary of the component of A_c containing c is a Jordan curve which we denote it by D_c. The map P_c^k on D_c is topologically conjugate to $\theta \mapsto d\theta$ on the unit circle. Therefore, there are $d-1$ fixed points of P_c^k on this Jordan curve which are repelling periodic points (of P_c) of period dividing period of $\bar{b}_c$ (its period can be strictly less than period of $\bar{b}_c$). Among all rays landing at these repelling periodic points, let θ_1 and θ_2 be the angles of the external rays bounding the sector containing the critical value of P_c (See Figure 3). The following theorem makes a connection between external rays R^{θ_1}, R^{θ_2} and the parameter external rays R_{θ_1}, R_{θ_2}. See [DH84] and [Sch04] for proofs.

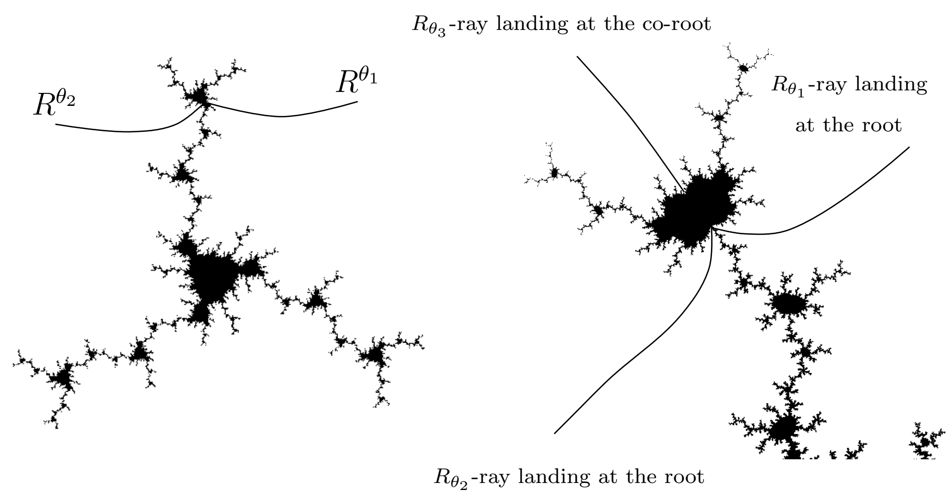

FIGURE 3. Figure on the left shows a primitively renormalizable Julia set and the external rays R^{θ_1} and R^{θ_2} landing at the corresponding repelling periodic point. Figure on the right is the corresponding primitive little multibrot copy. It also shows the parameter external rays R_{θ_1} and R_{θ_2} landing at the root point.

Theorem 2.3. *The parameter external rays R_{θ_1} and R_{θ_2} land at the root of $\mathscr{H}$ and moreover, these are the only rays that land at this point.*

Closure of the two parameter external rays R_{θ_1} and R_{θ_2} cut the parameter plane into two components. The one containing $\mathscr{H}$ with the root point attached to it is called the *wake* $W_{\mathscr{H}}$. So a wake is an open set with a point attached to its boundary. Given a wake $W_{\mathscr{H}}$ and an equipotential of level η, E_η, the *truncated wake* $W_{\mathscr{H}}(\eta)$ is the bounded component of $W_{\mathscr{H}} \setminus E_\eta$. Part of the connectedness locus contained in $W_{\mathscr{H}}$ is called the *limb* $\mathscr{L}_{\mathscr{H}}$ of the connectedness locus *originated* at $\mathscr{H}$. In other words, $\mathscr{L}_{\mathscr{H}} = W_{\mathscr{H}} \cap \mathcal{M}_d$. By definition, every limb is a closed set.

The wakes attached to the main hyperbolic component of $\mathcal{M}_d$ are called *primary wakes*. A limb associated to such a primary wake is called *primary limb*. If $\mathscr{H}$ is a hyperbolic component attached to the main hyperbolic component, all the wakes attached to $\mathscr{H}$ (except $W_{\mathscr{H}}$ itself) are called *secondary wakes*. Similarly, a limb associated to a secondary wake is called *secondary limb*. A *truncated limb* is obtained from a limb by removing a neighborhood of its root. Some secondary limbs are shown in Figure 2.

Given a parameter c in $\mathscr{H}$, we have the attracting cycle $\overline{b}_c$ as above and the associated repelling cycle $\overline{a}_c$ that is the landing point of the external rays R^{θ_1} and R^{θ_2}. The following result gives the dynamical meaning of the parameter values in the wake $W_{\mathscr{H}}$ bounded by parameter external rays R_{θ_1} and R_{θ_2} (See [Sch04] for further details).

Theorem 2.4. *For parameters c in $W_{\mathscr{H}} \setminus \{root\}$, the repelling cycle $\overline{a}_c$ stays repelling and moreover, the isotopic type of the ray portrait $\mathfrak{R}(\overline{a}_c)$ is fixed throughout $W_{\mathscr{H}}$.*

2.3. **Polynomial-like maps.** A polynomial-like map is a holomorphic proper branched covering of degree d, $f : U' \to U$, where U and U' are simply connected domains with U' compactly contained in U. For example, every polynomial can be viewed as a polynomial-like map once restricted to an appropriate neighborhood of the filled Julia set. This notion was introduced in [DH85b] to explain the presence of homeomorphic copies of the Mandelbrot set within the Mandelbrot set.

The filled Julia set $K(f)$ of a polynomial-like map f is naturally defined as

$$K(f) = \{z \in \mathbb{C} : f^n(z) \in U', \text{ for } n = 0, 1, 2, \dots\}.$$

The Julia set $J(f)$ is defined as the boundary of $K(f)$. These sets are connected if and only if $K(f)$ contains all critical points of f.

Two polynomial-like maps f and g are called *topologically* (*quasi-conformally*, *conformally*, *affinely*) *conjugate* if there are choices of domains U, U', V, and V' as well as a homeomorphism (quasi-conformal, conformal, or affine isomorphism, respectively) $h : U \to V$ such that $f : U' \to U$ and $g : V' \to V$ are polynomial-like maps and $h \circ f = g \circ h$ on U'.

Two polynomial like maps f and g are *hybrid or internally equivalent* if there is a quasi-conformal conjugacy (q.c. conjugacy for short) h between f and g such that $\overline{\partial} h = 0$ on $K(f)$. The following remarkable rigidity type theorem due to Douady and Hubbard [DH85b] states that the dynamics of a polynomial-like mapping is essentially the same as the one of a polynomial.

Theorem 2.5 (Straightening)**.** *Every polynomial-like map f is hybrid equivalent to (a suitable restriction of) a polynomial P of the same degree. Moreover, P is unique up to affine conjugacy when $K(f)$ is connected.*

In what follows we only consider polynomial-like maps with one branched point of degree d, assumed to be at zero by normalization, and refer to them as *unicritical polynomial-like* maps. By above theorem, any unicritical polynomial-like map with connected Julia set corresponds to a unique (up to affine conjugacy) unicritical polynomial $z \mapsto z^d + c$, with c in $\mathcal{M}_d$. Note that $z^d + c$ and $z^d + c/\lambda$ are conjugate via $z \mapsto \lambda z$ for every $d-1$th root of unity λ.

Given a polynomial-like map $f : U' \to U$, we can consider the *fundamental annulus* $A = U \setminus U'$. It is not canonic because any choice of $V' \Subset V$ such that $f : V' \to V$ is a polynomial-like map with the same Julia set gives a different annulus. However, we can associate a real number, *modulus of* f, to any polynomial-like map f as follows:

$$\operatorname{mod}(f) = \sup \operatorname{mod}(A)$$

where the sup is taken over all possible fundamental annuli A of f.

It is easy to see that the hybrid conjugacy in the straightening theorem is not unique. However, given a polynomial-like map, one can build a hybrid conjugacy with a uniform bound on its dilatation in terms of modulus of the polynomial-like map. This is used in an essentially way in the rest of this work. So we formulate it in the following proposition.

Proposition 2.6. *If $mod(f) \geq \mu > 0$, then one can choose a hybrid conjugacy in the straightening theorem with a bound on its dilatation in terms of μ.*

3. Modified principal nest

3.1. **Yoccoz puzzle pieces.** Recall that for a parameter $c \in \mathcal{M}_d$ outside of the main hyperbolic component, P_c has a unique dividing fixed point α_c. The $q \geq 2$ external rays $\mathfrak{R}(\alpha_c)$ landing at this fixed point together with an arbitrary equipotential E^r cut the domain inside E^r into q closed topological disks $Y_j^0, j = 0, 1, \ldots, q-1$, called *puzzle pieces of level zero*. That is, Y_j^0's are the closures of the bounded components of $\mathbb{C} \setminus \{E^r \cup \overline{\mathfrak{R}(\alpha_c)}\}$. The main property of this partition is that $P_c(\partial Y_j^0)$ does not intersect interior of any piece Y_i^0.

Puzzle pieces Y_i^n of level or depth n are defined as the closures of the connected components of $P_c^{-n}(\mathrm{int}(Y_j^0))$. They partition the region bounded by equipotential $P_c^{-n}(E^r)$ into finite number of closed disks. By definition, all puzzle pieces are bounded by piecewise analytic curves. A puzzle piece containing the critical point is referred to as *critical puzzle piece*. The *label* of each puzzle piece is the set of the angles of external rays bounding that puzzle piece. If the critical point does not land on the α_c-fixed point, there is a unique puzzle piece Y_0^n of every level n containing the critical point.

The family of all puzzle pieces of P_c of all levels has the following *Markov property*:

- Puzzle pieces are disjoint or nested. In the latter case, the puzzle piece of higher level is contained in the puzzle piece of lower level.
- Image of any puzzle piece of level $n \geqslant 1$ is a puzzle piece of level $n-1$. Moreover, $P_c : Y_j^n \to Y_k^{n-1}$ is a d-to-1 branched covering or univalent, depending on whether Y_j^n contains the critical point or not.

On the first level, there are $d(q-1)+1$ puzzle pieces organized as follows. The critical piece Y_0^1, $q-1$ pieces attached to the fixed point α_c that are denoted by Y_i^1, and $(d-1)(q-1)$ symmetric ones attached to $P_c^{-1}(\alpha_c) \setminus \{\alpha_c\}$ that are denoted by Z_i^1. Moreover, $P_c|Y_0^1$, d-to-1 covers Y_1^1, $P_c|Y_i^1$ univalently covers Y_{i+1}^1, for every $i = 1, \ldots, q-2$, and $P_c|Y_{q-1}^1$ univalently covers $Y_0^1 \cup \bigcup_{i=1}^{(d-1)(q-1)} Z_i^1$. Thus, $P_c^q(Y_0^1)$ truncated by $P_c^{-1}(E^r)$ is the union of Y_0^1 and Z_i^1's.

From now on we assume that $P_c^n(0) \neq \alpha_c$, for all n. Therefore, critical puzzle pieces of all levels are well defined. As it will be apparent in a moment, this condition is always the case for the parameters we are interested in.

3.2. **Favorite nest and renormalization.** Given a puzzle piece V containing 0, let $R_V : \text{Dom } R_V \subseteq V \to V$ denote *the first return map* to V. It is defined at every point z in V for which there exists a positive integer t with $P_c^t(z) \in \text{int } V$. So $R_V(z)$ is defined as $P_c^t(z)$ where t is the first positive moment when $P_c^t(z) \in \text{int } V$. Markov property of puzzle pieces implies that any component of Dom R_V is contained in V, and moreover, the restriction of this return map (P_c^t, for some t) to such a component is a d-to-1 or 1-to-1 proper map onto V. The component of Dom R_V containing the critical point is called the *central component* of R_V. If image of the critical point under the first return map belongs to the central component, the return is called *central return.*

The *first landing map* L_V to a puzzle piece $V \ni 0$ is defined at all points $z \in \mathbb{C}$ for which there exists an integer $t \geq 0$ with $P_c^t(z) \in \text{int } V$. It is the identity map on V, and univalently maps each component of Dom L_V onto V.

Consider a puzzle piece $Q \ni 0$. If the critical point returns back to Q under iteration of P_c, the central component $P \subset Q$ of R_Q is the pullback of Q by P_c^p along the orbit of the critical point, where p is the first moment when critical orbit enters int Q. Hence, $P_c^p : P \to Q$ is a proper map of degree d. This puzzle piece P is called the *first child* of Q.

The *favorite child* Q' of Q is constructed as follows; Let $p > 0$ be the first moment when $R_Q^p(0) \in \text{int}(Q \backslash P)$ (if it exists), and let $q > 0$ be the first moment (if it exists) when $R_Q^{p+q}(0) \in \text{int } P$ ($p + q$ is the moment of the first return back to P after the first escape of the critical point from P under iterate of R_Q). Now Q' is defined as the pullback of Q under R_Q^{p+q} containing the critical point. Markov property of puzzle pieces implies that the map $R_Q^{p+q} = P_c^k : Q' \to Q$ (for an appropriate $k > 0$) is a proper map of degree d. The main property of the favorite child is that the image of the critical point under $P_c^k : Q' \to Q$ belongs to the first child P.

Consider a unicritical polynomial P_c with q rays landing at its α-fixed point, and form the corresponding Yoccoz puzzle pieces. The map P_c is called *satellite renormalizable*, or *immediately renormalizable* if

$$P_c^{lq}(0) \in Y_0^1, \quad \text{for } l = 0, 1, 2, \ldots.$$

The map $P_c^q : Y_0^1 \to P_c^q(Y_0^1)$ is a proper branched covering of degree d. However, its domain is not compactly contained in its range. One can slightly enlarge Y_0^1 so that it is compactly contained in its range (see [Mil00a] for a detailed argument). Thus, P_c^q can be turned into a unicritical polynomial-like map. Note that the above condition

on the orbit of the critical point implies that the corresponding little Julia set is connected.

If P_c is not satellite renormalizable, then there is a first positive moment k such that $P_c^{kq}(0)$ belongs to some Z_i^1. Define Q^1 as the pullback of this Z_i^1 under P_c^{kq}. By the above process we form the first child P^1 and the favorite child Q^2 of Q^1. Repeating the above process we obtain a nest of puzzle pieces

$$Q^1 \supset P^1 \supset Q^2 \supset P^2 \supset \cdots \supset Q^n \supset P^n \supset \cdots \tag{3.1}$$

where P^i is the first child of Q^i, and Q^{i+1} is the favorite child of Q^i.

The above process stops if and only if one of the following happens:

- The map P_c is combinatorially non-recurrent, that is, the critical point does not return to some critical puzzle piece.
- Orbit of the critical point does not escape some first child P^n under iterate of R_{Q^n}, or equivalently, returns to all critical puzzle pieces of level bigger than some n are central.

In the first case, combinatorial rigidity of the critically non-recurrent parameters have been taken care of in [Mil00a]. In the latter case, $R_{Q^n} = P_c^k : P^n \to Q^n$ (for an appropriate k) is a unicritical polynomial-like map of degree d with P^n compactly contained in Q^n. The map P is called *primitively renormalizable* in this case. Note that the corresponding little Julia set is connected because all returns of the critical point to Q^n are central by definition.

A unicritical polynomial is called *renormalizable* if it is satellite or primitively renormalizable.

3.3. **Complex bounds and pseudo-conjugacies.** The general strategy, starting with Yoccoz's work on quadratics [Hub93], to prove local connectivity of Julia sets and rigidity of complex unicritical polynomials has been to show the shrinking of nest of puzzle pieces to points. To deal with non-renormalizable and recurrent polynomials, the following *a priori* bounds property has been proved in [AKLS05].

Theorem 3.1. *There exists a constant $\delta > 0$ such that for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) > 0$, with the following property. In the nest of puzzle pieces* (3.1), *if* $\operatorname{mod}(Q^1 \setminus P^1) > \varepsilon$ *then for all $n \geq n_0$ we have* $\operatorname{mod}(Q^n \setminus P^n) > \delta$.

If P_c is combinatorially recurrent, the critical point does not land at α-fixed point. Therefore, puzzle pieces of all levels are well defined. The *combinatorics of P_c* up to level n is defined as an equivalence relation on the set of angles of puzzle pieces up to level n. Two angles θ_1 and θ_2 are equivalent if the corresponding rays R^{θ_1} and R^{θ_2} land

at the same point. One can see that the combinatorics of a map up to level $n+t$ determines the puzzle piece Y_j^n of level n containing the critical value $P_c^t(0)$ for all positive integers n and t. Two non-renormalizable maps are called *combinatorially equivalent* if they have the same combinatorics up to every level n, that is, they have the same set of labels of puzzle pieces and the same equivalence relation on them. Combinatorics of a renormalizable map will be defined in Section 3.4.

Two unicritical polynomials P_c and $P_{\tilde{c}}$ with the same combinatorics up to level n are *pseudo-conjugate up to level* n if there is an orientation preserving homeomorphism $H : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, such that $H(Y_j^0) = \widetilde{Y}_j^0$, for all j, and $H \circ P_c = P_{\tilde{c}} \circ H$ outside of the critical puzzle piece Y_0^n. A pseudo-conjugacy H is said to *match the Böttcher marking*, if near infinity it becomes identity in the Böttcher coordinates for P_c and $P_{\tilde{c}}$. Thus, by equivariance property of a pseudo-conjugacy, it is the identity map in the Böttcher coordinates outside of $\cup_j Y_j^n$.

Let q_m and p_m denote the levels of the puzzle pieces Q^m and P^m in the nest 3.1, that is, $Q^m = Y_0^{q_m}$, and $P^m = Y_0^{p_m}$. The following statement is the main technical result of [AKLS05] which is frequently used in the proof of our main theorem.

Theorem 3.2. *Assume that a nest of puzzle pieces*

$$Q^1 \supset P^1 \supset Q^2 \supset P^2 \supset \cdots \supset Q^m \supset P^m \tag{3.2}$$

is obtained for P_c. If $P_{\tilde{c}}$ is combinatorially equivalent to P_c up to level q_m, where $Q^m = Y_0^{q_m}$, then there exists a K-q.c. pseudo-conjugacy H up to level q_m between P_c and $P_{\tilde{c}}$ which matches the Böttcher marking.

To control the dilatation of the pseudo-conjugacy obtained in this theorem, we show the following statement.

Proposition 3.3. *Assume that the nest of puzzle pieces in the above theorem is defined using equipotential of level η, then dilatation of the q.c. pseudo-conjugacy obtained in that theorem depends only on the hyperbolic distance between c and $\tilde{c}$ in the primary wake truncated by parameter equipotential of level η and modulus of the annulus $Q^1 \setminus P^1$.*

Proof. To prove the proposition, we need a brief sketch of the proof of the above theorem. For more details one may refer to [AKLS05].

Combinatorial equivalence of P_c and $P_{\tilde{c}}$ up to level zero implies that the parameters c and $\tilde{c}$ belong to the same truncated wake $W(\eta)$ attached to the main component of the connectedness locus. Inside $W(\eta)$, the q external rays $\mathfrak{R}(\alpha)$ and the equipotential $E(h)$, for any $h > \eta$, move holomorphically in $\mathbb{C} \setminus 0$. That is, there exists a holomorphic

motion $\Phi : W(\eta) \times \{\mathfrak{R}(\alpha) \cup E(h)\} \to W(\eta) \times \mathbb{C}$, given by $B_{\tilde{c}}^{-1} \circ B_c$ in the second coordinate, such that

$$\Phi(c, \mathfrak{R}(\alpha) \cup E(h)) = (\tilde{c}, \mathfrak{R}(\widetilde{\alpha}) \cup \widetilde{E}(h)).$$

Outside of equipotential $E(h)$, this holomorphic motion extends to a motion holomorphic in both variables (c, z) which is obtained from the Böttcher coordinates near ∞. By [Slo91] the map $\Phi(\tilde{c}, \cdot) \circ \Phi(c, \cdot)^{-1}$ extends to a K_0 q.c. map $G_0 : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, where K_0 depends only on the hyperbolic distance between c and $\tilde{c}$ in $W(\eta)$. This gives a q.c. map $G_0 : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ which conjugates P_c and $P_{\tilde{c}}$ outside of puzzle pieces of level zero.

By adjusting the q.c. map G_0 inside equipotential $E(h)$ such that it sends c to $\tilde{c}$, we get a q.c. map (not necessarily with the same dilatation) G'_0. By lifting G'_0 via P_c and $P_{\tilde{c}}$ we obtain a new q.c. map G_1. Repeating this process, for $i = 1, 2, \dots, n = q_m$, which is adjusting the q.c. map G_i inside union of puzzle pieces of level $i+1$ so that it sends c to $\tilde{c}$ and then lifting it via P_c and $P_{\tilde{c}}$, we obtain a q.c. map G_{i+1} (not with the same dilatation) conjugating P_c and $P_{\tilde{c}}$ outside union of puzzle pieces of level $i+1$. At the end, we have a q.c. map G_n which conjugates P_c and $P_{\tilde{c}}$ outside of equipotential $E(h/d^n)$.

The nest of puzzle pieces

$$\widetilde{Q}^1 \supset \widetilde{P}^1 \supset \widetilde{Q}^2 \supset \widetilde{P}^2 \supset \cdots \supset \widetilde{Q}^m \supseteq \widetilde{P}^m$$

for $P_{\tilde{c}}$ is defined as the image of the nest (3.2) under G_n. Combinatorial equivalence of P_c and $P_{\tilde{c}}$ implies that this new nest has the same properties as the one for P_c. In other words, $\widetilde{Q}^{i+1}$ is the favorite child of $\widetilde{Q}^i$, and $\widetilde{P}^i$ is the first child of $\widetilde{Q}^i$. Hence, Theorem 3.1 applies to this nest as well. By properties of these nests, one constructs a K-q.c. map H_n from the critical puzzle piece Q^n to $\widetilde{Q}^n$, where K only depends on the *a priori* bounds δ and the hyperbolic distance between c and $\tilde{c}$ in $W(\eta)$. The pseudo-conjugacy H_n is obtained from univalent lifts of H_n onto other puzzle pieces. □

If P_c is renormalizable, the process of constructing modified principal nest stops at some level and all returns to the critical puzzle pieces of higher level are central. One can see that the critical puzzle pieces do not shrink to 0.

3.4. **Combinatorics of a map.** If a map P_{c_0} is renormalizable, there is a maximal homeomorphic copy $\mathcal{M}_d^1 \ni c_0$ of the connectedness locus within the connectedness locus satisfying the following properties (see[DH85b]): For $c \in \mathcal{M}_d^1 \setminus \{\text{root}\}$, $P_c : z \mapsto z^d + c$ is renormalizable, and in addition, there is a holomorphic motion of the dividing fixed

point α_c and the rays landing at it on a neighborhood of $\mathcal{M}_d^1 \setminus \{\text{root}\}$, such that the renormalization of P_c is associated to this fixed point and external rays. Furthermore, all parameters in this copy have Yoccoz puzzle pieces of all levels with the same labels. It is maximal in a sense that it is not contained in any other copy except the actual connectedness locus. This homeomorphism, from the copy to the connectedness locus, is not unique because of the symmetry in the connectedness locus. However, we make it unique by sending the only root point of the copy to the landing point of the parameter external ray of angle 0. We denote this first renormalization of P_c by $\mathcal{R}P_c$.

Assume that $\mathcal{R}P_c$ is equal to $P_c^j : U \to U'$, for some positive integer j and a topological disk U compactly contained in U'. By straightening theorem, $\mathcal{R}P_c$ is conjugate to a unicritical polynomial $P_{c'}$. The polynomial $P_{c'}$ is determined up to conformal equivalence in this theorem. However, there are only $d-1$ polynomials in each conformal class (i.e., $c' \cdot \lambda$, for $\lambda^{d-1} = 1$). We make this parameter unique by choosing the image of c under the unique homeomorphism determined above.

If $P_{c'}$ is also renormalizable, P_c is called *twice renormalizable*. Let integer $k > 1$, and topological disks V and V' be such that $P_{c'}^k : V \to V'$ is the first renormalization of $P_{c'}$ determined as above. Define $\widetilde{V}$ and $\widetilde{V}'$ as χ-preimage of V and V', respectively, where χ is a straightening of $\mathcal{R}P_c$. One can see that χ conjugates $P_c^{jk} : \widetilde{V} \to \widetilde{V}'$ with $P_{c'}^k : V \to V'$. Therefore, $P_c^{jk} : \widetilde{V} \to \widetilde{V}'$ is also a polynomial-like map. We denote this map by $\mathcal{R}^2 P_c$.

Above process may be continued to associate a finite or infinite sequence $P_c, \mathcal{R}P_c, \mathcal{R}^2 P_c, \ldots$, of polynomial-like maps to P_c, and accordingly, call P_c *at most finitely* or *infinitely renormalizable*. Let $P_{c_1}, P_{c_2}, P_{c_3}, \ldots$ denote the polynomials obtained form straightening the polynomial-like maps $\{\mathcal{R}^n P_c\}_{n=0}$. We define the finite or infinite sequence

$$\tau(P_c) := \langle \mathcal{M}_d^1, \mathcal{M}_d^2, \ldots \rangle,$$

of maximal copies of the locus associated to P_c, where $\mathcal{M}_d^n$ corresponds to the renormalization $\mathcal{R}P_{c_{n-1}}$. Earlier in Section 3.3 we defined the combinatorics of a non-renormalizable unicritical polynomial as the equivalence relation on the labels of the Yoccoz puzzle pieces. It turns out that all parameters in a given copy of the connectedness locus within the parameter space have the same combinatorics in this sense. To further refine our definition of the combinatorics, one may consider the same equivalence relation, landing at the same points, on a larger set of angles. For an infinitely renormalizable P_c, the sequence $\tau(P_c)$ is called the *combinatorics* of P_c. This definition, which is chosen at our

convenience, is equivalent to the above equivalence relation on the set of angles of all periodic external rays.

Hence, two infinitely renormalizable maps are called *combinatorially equivalent* if they have the same combinatorics, i.e., correspond to the same sequence of maximal connectedness locus copies.

We say an infinitely renormalizable P_c satisfies the *secondary limbs condition*, if all the parameters $c_1, c_2, \ldots$, obtained from straightening the maps $\{\mathcal{R}^n P_c\}_{n=0}^{\infty}$ belong to a finite number of truncated secondary limbs. Let $\mathcal{SL}$ stand for the class of infinitely renormalizable unicritical polynomial-like maps satisfying the secondary limbs condition.

An infinitely renormalizable map P_c satisfies *a priori bounds*, if there exists an $\varepsilon > 0$ with $\operatorname{mod}(\mathcal{R}^m P_c) \geq \varepsilon$, for all $m \geq 1$.

4. Proof of the rigidity theorem

4.1. **Reductions.**

Theorem 4.1 (Rigidity theorem). *Let f and $\tilde{f}$ be two infinitely renormalizable unicritical polynomial-like maps satisfying $\mathcal{SL}$ and the a priori bounds conditions. If f and $\tilde{f}$ are combinatorially equivalent, then they are hybrid equivalent.*

Remark. In particular if the two maps f and $\tilde{f}$ in the above theorem are polynomials, then hybrid equivalence becomes conformal equivalence. That is because the Böttcher coordinate, which conformally conjugates the two maps on the complement of the Julia sets, can be glued to the hybrid conjugacy on the Julia set. See Proposition 6 in [DH85b] for a precise proof of this.

The proof of the rigidity theorem breaks into following steps:

$$\begin{array}{c}\text{combinatorial equivalence}\\ \Downarrow \\ \text{topological equivalence}\\ \Downarrow \\ \text{q.c. equivalence}\\ \Downarrow \\ \text{hybrid equivalence}\end{array}$$

It has been shown in [Jia00] that any *unbranched* infinitely renormalizable map with *a priori* bounds has locally connected Julia set. A renormalization $f^n : U \to V$ is called unbranched if the domains U and V which provide the *a priori* bounds also satisfy

$$\mathcal{PC}(f) \cap U = \mathcal{PC}(f^n : U \to V).$$

Here, unbranched condition follows from our combinatorial condition and *a priori* bounds (see [Lyu97] Lemma 9.3). Then, the first step, topological equivalence of combinatorially equivalent maps, follows from local connectivity of the Julia sets by the Carathéodory's theorem. That is, The identity map in the Böttcher coordinates extends onto Julia set. Indeed by [Dou93] there is a topological model for the Julia set of these maps based on their combinatorics.

The last step, only under the a priori bounds condition, follows from McMullen's rigidity theorem [McM94] (Theorem 10.2). He has shown that an infinitely renormalizable quadratic polynomial-like map with *a priori* bounds does not have any nontrivial invariant line field on its Julia set. The same proof works for degree d unicritical polynomial-like maps. It follows that any q.c. conjugacy h between f and $\tilde{f}$ satisfies $\overline{\partial} h = 0$ almost everywhere on the Julia set. Therefore, h is a hybrid conjugacy between f and $\tilde{f}$. However, if all infinitely renormalizable unicritical maps in a given combinatorial class satisfy the *a priori* bounds condition, it is easier to show that q.c. conjugacy implies hybrid conjugacy for that class rather than showing that there is no nontrivial invariant line field on the Julia set. Since we are going to apply our theorem to combinatorial classes for which *a priori* bounds have been established, we will prove this in Proposition 4.16.

So assume that f and $\tilde{f}$ are topologically conjugate. We want to show the following:

Theorem 4.2. *Let f and $\tilde{f}$ be infinitely renormalizable unicritical polynomial-like maps satisfying the a priori bounds and $\mathcal{SL}$ conditions. If f and $\tilde{f}$ are topologically conjugate then they are q.c. conjugate.*

Given sets $A \subseteq B \subseteq C$ and $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$, the notation $h : (C, B, A) \to (\widetilde{C}, \widetilde{B}, \widetilde{A})$ means that h is a map from C to $\widetilde{C}$ with $h(B) \subseteq \widetilde{B}$ and $h(A) \subseteq \widetilde{A}$.

4.2. **Thurston equivalence.** Suppose two unicritical polynomial-like maps $f : U_2 \to U_1$ and $\tilde{f} : \widetilde{U}_2 \to \widetilde{U}_1$ are topologically conjugate. A q.c. map

$$h : (U_1, U_2, \mathcal{PC}(f)) \to (\widetilde{U}_1, \widetilde{U}_2, \mathcal{PC}(\tilde{f}))$$

is a *Thurston conjugacy* if it is homotopic to a topological conjugacy

$$\psi : (U_1, U_2, \mathcal{PC}(f)) \to (\widetilde{U}_1, \widetilde{U}_2, \mathcal{PC}(\tilde{f}))$$

between f and $\tilde{f}$ relative $\partial U_1 \cup \partial U_2 \cup \mathcal{PC}(f)$. Note that a Thurston conjugacy is not a conjugacy between the two maps. It is a conjugacy on the postcritical set and homotopic to a conjugacy on the complement

of the postcritical set. We see in the next lemma that it is in the "right" homotopy class.

The following result is due to Thurston and Sullivan [Sul92] which originates the "pull-back method" in holomorphic dynamics.

Lemma 4.3. *Thurston conjugate unicritical polynomial-like maps are q.c. conjugate.*

Proof. Assume that

$$h_1 : (U_1, U_2, \mathcal{PC}(f)) \to (\widetilde{U}_1, \widetilde{U}_2, \mathcal{PC}(\tilde{f}))$$

is a Thurston conjugacy homotopic to a topological conjugacy

$$\Psi : (U_1, U_2, \mathcal{PC}(f)) \to (\widetilde{U}_1, \widetilde{U}_2, \mathcal{PC}(\tilde{f}))$$

relative $\partial U_1 \cup \partial U_2 \cup \mathcal{PC}(f)$.

As $f : U_2 \setminus \{0\} \to U_1 \setminus \{f(0)\}$ and $\tilde{f} : \widetilde{U}_2 \setminus \{0\} \to \widetilde{U}_1 \setminus \{\tilde{f}(0)\}$ are covering maps, $h_1 : U_1 \setminus \{f(0)\} \to \widetilde{U}_1 \setminus \{\tilde{f}(0)\}$ can be lifted to a homeomorphism $h_2 : U_2 \setminus \{0\} \to \widetilde{U}_2 \setminus \{0\}$. Moreover, since h_1 satisfies the *equivariance relation* $h_1 \circ f = \tilde{f} \circ h_1$ on the boundary of U_2, h_2 can be extended onto $U_1 \setminus U_2$ by h_1. It also extends to the critical point of f by sending it to the critical point of $\tilde{f}$. Let us denote this new map by h_2. For the same reason, every homotopy h_t between Ψ and h_1 can be lifted to a homotopy between Ψ and h_2. As f and $\tilde{f}$ are holomorphic maps, h_2 has the same dilatation as dilatation of h_1. This implies that the new map h_2 is also a Thurston conjugacy with the same dilatation as the one of h_1. By definition, the new map h_2 satisfies the equivariance relation on the annulus $U_2 \setminus f^{-1}(U_2)$.

Repeating the same process with h_2, we obtain a q.c. map h_3 and so on. Thus, we have a sequence of K-q.c. maps h_n from U_1 to $\widetilde{U}_1$ which satisfies the equivariance relation on the annulus $U_2 \setminus f^{-n}(U_2)$. All these maps can be extended onto complex plane with a uniform bound on their dilatation. This family of q.c. maps is normalized at points $0, f(0), f^2(0), \dots$ by mapping them to the corresponding points $0, \tilde{f}(0), \tilde{f}^2(0), \dots$. Compactness of this class of maps implies that there is a subsequence h_{n_j} that converges to a K-q.c. map H on U_1.

For every z outside of the Julia set, the sequence $h_{n_j}(z)$ stabilizes and, by definition, eventually $h_{n_j} \circ f(z) = f \circ h_{n_j}(z)$. Taking limit of both sides, we obtain $H \circ f(z) = f \circ H(z)$ for every such z. As filled Julia set of an infinitely renormalizable unicritical map has empty interior, conjugacy relation for an arbitrary z on the Julia set follows from continuity of H. □

By the *a priori* bounds assumption in the theorem, there are topological disks $V_{n,0} \Subset U_{n,0}$ containing 0 such that $\mathcal{R}^n f := f^{t_n} : V_{n,0} \to U_{n,0}$ is a unicritical degree d polynomial-like map and $\mod (U_{n,0} \backslash V_{n,0}) \geq \varepsilon$. By going several levels down, i.e., considering $f^{t_n} : f^{-kt_n}(V_n) \to f^{-kt_n}(U_n)$ for some positive integer k, we may assume that $\mod (U_n \setminus V_n)$ and $\mod (\widetilde{U}_n \setminus \widetilde{V}_n)$ are proportional. Also by slightly shrinking the domains, if necessary, we may assume that these domains have smooth boundaries. Hence, we can assume the following:

– There exist positive constants ε and η such that for every $n \geq 1$, we have
$$\varepsilon \leq \mod (U_{n,0} \setminus V_{n,0}) \leq \eta,$$
$$\varepsilon \leq \mod (\widetilde{U}_{n,0} \setminus \widetilde{V}_{n,0}) \leq \eta,$$
– For every $n \geq 1$, $U_{n,0}$ and $\widetilde{U}_{n,0}$ have smooth boundaries.

Note that the above condition implies that there exits a constant M such that for every $n \geq 1$, we have
$$\frac{1}{M} \leq \frac{\mod (U_n \setminus V_n)}{\mod (\widetilde{U}_n \setminus \widetilde{V}_n)} \leq M.$$

We use the following notations throughout the rest of this note.

$f : V_0 \to U_0, J_{0,0} = K_{0,0} = K(f) = J(f),$
$\mathcal{R}f = f^{t_1} : V_{1,0} \to U_{1,0}, J_{1,0} = K_{1,0} = K(\mathcal{R}f),$
$\mathcal{R}^2 f = f^{t_2} : V_{2,0} \to U_{2,0}, J_{2,0} = K_{2,0} = K(\mathcal{R}^2 f),$
$\vdots$
$\mathcal{R}^n f = f^{t_n} : V_{n,0} \to U_{n,0}, J_{n,0} = K_{n,0} = K(\mathcal{R}^n f),$
$\vdots$

The domain $V_{n,i}$, for $i = 1, 2, \ldots, t_n - 1$, is defined as the pullback of $V_{n,0}$ under f^{-i} containing the *little Julia set* $J_{n,i} := f^{t_n - i}(J_{n,0})$. Similarly, $U_{n,i}$ is defined as the component of $f^{-i}(U_{n,0})$ containing $V_{n,i}$ so that $f^{t_n} : V_{n,i} \to U_{n,i}$ is a polynomial-like map. The domain $W_{n,i}$ is defined as the preimage of $V_{n,i}$ under the map $f^{t_n} : V_{n,i} \to U_{n,i}$.

Accordingly, $K_{n,i}$ is defined as the component of $f^{-i}(K_{n,0})$ inside $V_{n,i}$. Note that $\mathcal{R}^n f : V_{n,i} \to U_{n,i}$ is a polynomial-like map with the filled Julia set $K_{n,i}$ and is conjugate to $\mathcal{R}^n f : V_{n,0} \to U_{n,0}$ by conformal isomorphism $f^i : U_{n,i} \to U_{n,0}$.

It has been proved in [Lyu97] (Lemma 9.2) that for parameters under our assumption there is always definite space in between little Julia sets in the primitive case. Compare with our proof of Lemma 4.9. Definite space between little Julia sets guarantees that there exist choices of domains $U_{n,i}$ which are disjoint for different i's and the annuli $U_{n,i} \backslash V_{n,i}$

have definite moduli. So we will assume that on the primitive levels, the domains $U_{n,i}$ are disjoint for different i's.

In all of the above notation, the first lower subscripts denote the level of renormalization and the second lower subscripts run over little filled Julia sets, Julia sets and their neighborhoods accordingly. In what follows all corresponding objects for $\tilde{f}$ will be marked with a tilde and any notation introduced for f will be automatically introduced for $\tilde{f}$ as well.

To build a Thurston conjugacy, we first introduce multiply connected domains $\Omega_{n(k),i}$ (and $\widetilde{\Omega}_{n(k),i}$) in $\mathbb{C}$ for an appropriate subsequence $n(1) < n(2) < n(3) < \cdots$ of the renormalization levels and a sequence of q.c. maps with uniformly bounded distortion

$$h_{n(k),i} : \Omega_{n(k),i} \to \widetilde{\Omega}_{n(k),i}$$

for $k = 0, 1, 2, \dots$ and $i = 0, 1, 2, \dots, t_{n(k)} - 1$. These domains will satisfy the following properties:

- Each $\Omega_{n(k),j}$, for $n(k) \geq 0$, is a topological disk minus $\frac{t_{n(k+1)}}{t_{n(k)}}$ topological disks $D_{n(k+1),j+it_{n(k)}}$, for $i = 0, 1, \dots, t_{n(k+1)}/t_{n(k)}-1$.
- Each $\Omega_{n(k),i}$, for $n(k) \geq 1$, is well inside $D_{n(k),i}$ which means that the moduli of the annuli obtained from $D_{n(k),i} \setminus \Omega_{n(k),i}$ are uniformly bounded below for all $n(k)$ and i.
- Every *little postcritical set* $J_{n(k),i} \cap \mathcal{PC}(f)$ is well inside $D_{n(k),i}$.
- Every $D_{n(k),i}$ is the pullback of $D_{n(k),0}$ under f^{-i} containing $J_{n(k),i} \cap \mathcal{PC}(f)$, and every $\Omega_{n(k),i}$ is the component of $f^{-i}(\Omega_{n(k),0})$ inside $D_{n(k),i}$.

Finally, we construct a Thurston conjugacy by appropriately gluing the maps $h_{n(k),i} : \Omega_{n(k),i} \to \widetilde{\Omega}_{n(k),i}$ together on the complement of all these multiply connected domains (which is a union of annuli). See Figure 4.

4.3. **The domains $\Omega_{n,j}$ and the maps $h_{n,j}$.** By applying the straightening theorem to the polynomial-like maps

$$\mathcal{R}^{n-1} f : V_{n-1,0} \to U_{n-1,0}$$

we get $K_1(\varepsilon)$-q.c. maps and unicritical polynomials $\mathbf{f}_{c_{n-1}}$, such that

$$S_{n-1} : (U_{n-1,0}, V_{n-1,0}, 0) \to (\Upsilon^0_{n-1}, \Upsilon^1_{n-1}, 0), \quad S_{n-1} \circ \mathcal{R}^{n-1} f = \mathbf{f}_{c_{n-1}} \circ S_{n-1}. \tag{4.1}$$

See Figure 5.

Remark. To make our notations easier to follow, we will drop the second subscript whenever it is zero and it does not create any confusion.

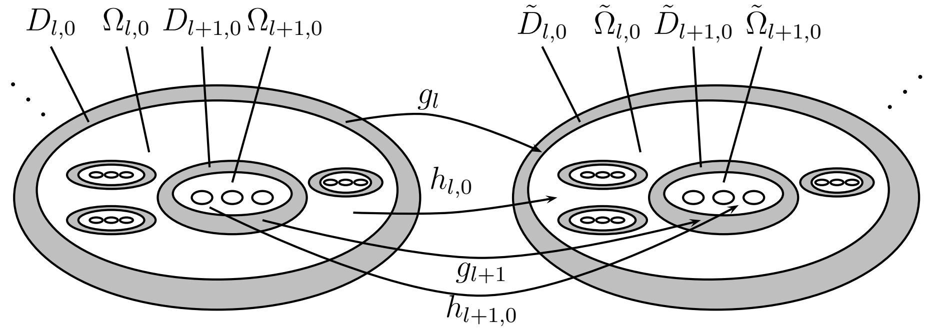

FIGURE 4. The multiply connected domains and the buffers

Also, all objects on the dynamic planes of $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$ (the ones after straightening) will be denoted by **bold** face of notations used for objects on the dynamic planes of f and $\tilde{f}$.

To define $\Omega_{n-1,j}$ and $h_{n-1,j}$, because of the difference in type of renormalizations, we will consider the following three cases:

$\mathscr{A}$. $\mathcal{R}^{n-1}f$ is primitively renormalizable.
$\mathscr{B}$. $\mathcal{R}^{n-1}f$ is satellite renormalizable and $\mathcal{R}^n f$ is primitively renormalizable.
$\mathscr{C}$. Both $\mathcal{R}^{n-1}f$ and $\mathcal{R}^n f$ are satellite renormalizable.

For a given infinitely renormalizable map f_c, the renormalization on each level is of primitive or satellite type. Therefore, we can associate a word

$$P \dots PS \dots SP \dots \tag{4.2}$$

of P and S where a P or a S in the i's place means that the i's renormalization of f_c is of primitive or satellite type, respectively. Corresponding to any such word, we define a word of cases $\mathscr{A}^{m_1}\mathscr{B}^{m_2}\mathscr{C}^{m_3}\dots$, with non-negative integers m_j, as follows. Inductively, starting from left, a P is replaced by $\mathscr{A}$, SP by $\mathscr{B}$, and SS by $\mathscr{C}S$. By repeating this process, we obtain a word of cases which is used to decide which case to pick at each step. More precisely, for a word of cases $\mathscr{A}^{m_1}\mathscr{B}^{m_2}\mathscr{C}^{m_3}\dots$, with non-negative integers m_j, obtained for a parameter we repeat Case $\mathscr{A}$, m_1 times then repeat Case $\mathscr{B}$, m_2 times and so on. This will also introduce the sequence $n(k)$, for $k = 1, 2, 3, \dots$ as follows. Given the word (4.2) corresponding to a parameter c as above, the sequence $n(k)$

is obtained from the sequence of natural numbers $1, 2, 3, \dots$ by removing all the integers l for which there is a S in the $l-1$th place and a P in the lth place. That is, we skip level of any primitive renormalization occurring after a satellite one.

Case $\mathscr{A}$: We need the following lemma to show that there are equipotentials of sufficiently high level $\eta(\varepsilon)$ inside the domains $S_{n-1}(W_{n-1,0})$ and $\widetilde{S}_{n-1}(\tilde{W}_{n-1,0})$ in the dynamic planes of the maps $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$.

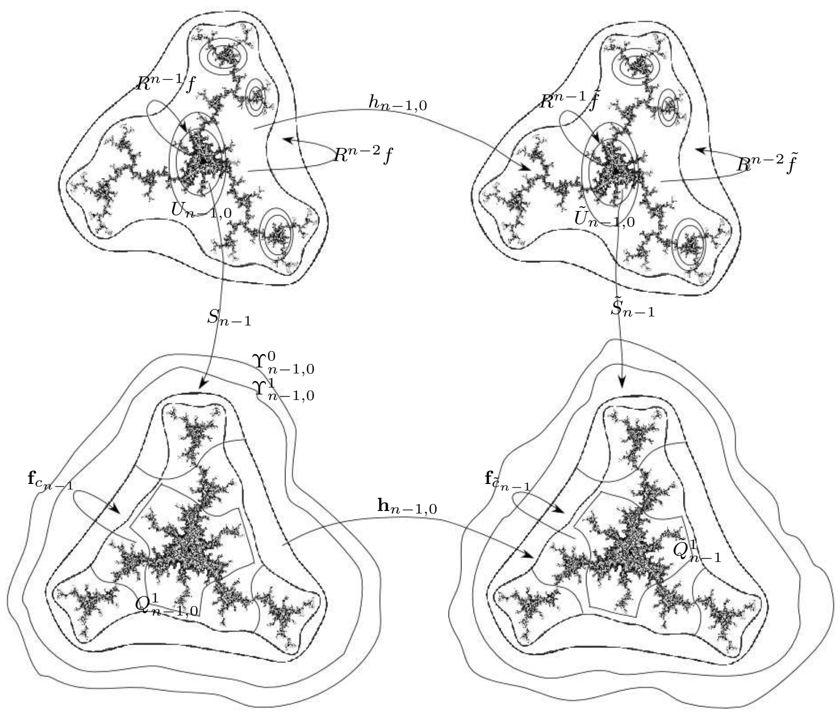

FIGURE 5. Primitive case

Lemma 4.4. *If $P_c : U' \to U$ is a unicritical polynomial with connected Julia set and* mod $(U \setminus U') \geq \varepsilon$*, then U' contains equipotentials of level less than $\eta(\varepsilon)$ depending only on ε.*

Proof. The map P_c on the complement of $K(P_c)$ is conjugate to P_0 on the complement of the closed unit disk D_1 by Böttcher coordinate B_c. Since levels of equipotentials are preserved under this map, and modulus is conformal invariant, it is enough to prove the statement for $P_0 : V' \to V$, for V' compactly contained in V and mod $(V \setminus V') \geq \varepsilon$. As $P_0 : P_0^{-1}(V \setminus V') \to (V \setminus V')$ is a covering of degree d, modulus of

the annulus $P_0^{-1}(V \setminus V')$ is at least ε/d which implies that modulus of $V' \setminus D_1 \geq \varepsilon/d$. By Grötzsch problem in [Ahl06] (Section A in Chapter III) we conclude that $V' \setminus D_1$ contains a round annulus $D_{\eta(\varepsilon)} \setminus D_1$. $\square$

By considering the equipotentials of level $\eta(\varepsilon)$ contained in S_{n-1} of $W_{n-1,0}$ and $\widetilde{S}_{n-1}$ of $\tilde{W}_{n-1,0}$, obtained in the previous lemma, and the external rays landing at the dividing fixed points α_{n-1} and $\tilde{\alpha}_{n-1}$ of the maps $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$, we can form the favorite nest of puzzle pieces (3.1) introduced in Section 3.2.

Let $Q_{n,0}^{\chi_n} := Y_{n,0}^{q_{\chi_n}}$ and $P_{n,0}^{\chi_n}$ denote the last critical puzzle pieces obtained in the nest (3.1), and

$$\mathbf{h}_{n-1} := \mathbf{h}_{n-1,0} : \mathbb{C} \to \mathbb{C},$$

denote the corresponding K_2-q.c. pseudo-conjugacy obtained in Theorem 3.2. The hyperbolic distance between the parameters c_{n-1} and $\tilde{c}_{n-1}$ in the truncated primary wake containing c_{n-1} and $\tilde{c}_{n-1}$, $W(\eta(\varepsilon))$, is bounded by some $M(\varepsilon)$ depending only on ε and the combinatorial class $\mathcal{SL}$. That is because c_{n-1} and $\tilde{c}_{n-1}$ belong to a finite number of truncated limbs which is a compact subset of $W(\eta(\varepsilon))$. It has been proved in [Lyu97], Theorem I, that modulus of the top annulus $Q^1 \setminus P^1$ is uniformly bounded below depending only on the combinatorial class $\mathcal{SL}$. The same proof, based one continuous dependence of certain rays on the parameter and compactness of the class $\mathcal{SL}$, works for unicritical polynomials as well. Therefore, by Proposition 3.3, dilatation of the pseudo-conjugacy obtained in Theorem 3.2 is uniformly bounded by some constant K_2 depending only on ε and $\mathcal{SL}$.

Lets denote the components of $\mathbf{f}_{c_{n-1}}^{-i}(Q_{n,0}^{\chi_n})$ and $\mathbf{f}_{c_{n-1}}^{-i}(P_{n,0}^{\chi_n})$ containing the little Julia sets $J_{n,i}$, for $i = 0, 1, 2, \dots, t_n/t_{n-1}-1$, by $Q_{n,i}^{\chi_n}$ and $P_{n,i}^{\chi_n}$, respectively. Note that t_n/t_{n-1} is the period of the first renormalization of $\mathbf{f}_{c_{n-1}}$.

As the polynomials $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$ also satisfy our combinatorial condition and the *a priori* bounds assumption, there is a topological conjugacy, denoted by $\boldsymbol{\psi}_{n-1}$, between them obtained from extending $B_{\tilde{c}_{n-1}}^{-1} \circ B_{c_{n-1}}$ onto the Julia set.

Now we would like to adjust $\mathbf{h}_{n-1} : Q_n^{\chi_n} \to \widetilde{Q}_n^{\chi_n}$, using the dynamics of the maps

$$\mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}} : P_n^{\chi_n} \to Q_n^{\chi_n} \text{ and } \mathbf{f}_{\tilde{c}_{n-1}}^{t_n/t_{n-1}} : \widetilde{P}_n^{\chi_n} \to \widetilde{Q}_n^{\chi_n},$$

to have equivariance property on a larger set. Let A_n^0 denote the closure of the annulus $Q_n^{\chi_n} \setminus P_n^{\chi_n}$, and A_n^k, for $k = 0, 1, 2, \dots$, denote the component of $\mathbf{f}_{c_{n-1}}^{-kt_n/t_{n-1}}(A_n^0)$ around $J_{n,0}$. We can lift $\mathbf{h}_{n-1}$ via

$\mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}} : Q_n^{\chi_n} \to \mathbb{C}$ and $\mathbf{f}_{\tilde{c}_{n-1}}^{t_n/t_{n-1}} : \widetilde{Q}_n^{\chi_n} \to \mathbb{C}$ to obtain a K_2-q.c. map $g : A_n^0 \to \widetilde{A}_n^0$ which is homotopic to $\boldsymbol{\psi}_{n-1}$ relative ∂A_n^0. That is because by the external rays connecting $\partial P_n^{\chi_n}$ to $\partial Q_n^{\chi_n}$, the annulus A_n^0 is partitioned into some topological disks and the two maps coincide on the boundaries of these topological disks.

As $\mathbf{f}_{c_{n-1}}^{kt_n/t_{n-1}} : A_n^k \to A_n^0$ and $\mathbf{f}_{\tilde{c}_{n-1}}^{kt_n/t_{n-1}} : \widetilde{A}_n^k \to \widetilde{A}_n^0$ are holomorphic unbranched coverings, g can be lifted to a K_2-q.c. map from A_n^k to $\widetilde{A}_n^k$, for every $k \geq 1$. All these lifts are the identity map in the Böttcher coordinates on the boundaries of these annuli. Hence, they match together to K_2-q.c. conjugate the two maps

$$\mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}} : P_n^{\chi_n} \setminus J_n \to Q_n^{\chi_n} \setminus J_n \text{ and } \mathbf{f}_{\tilde{c}_{n-1}}^{t_n/t_{n-1}} : \widetilde{P}_n^{\chi_n} \setminus \widetilde{J}_n \to \widetilde{Q}_n^{\chi_n} \setminus \widetilde{J}_n.$$

Finally, we would like to extend this map further onto little Julia set J_n. This is a special case of a more general argument presented below.

Given a polynomial f with connected filled Julia set $K(f)$, the *rotation* of angle θ on $\mathbb{C} \setminus K(f)$ is defined as the rotation of angle θ in the Bötcher coordinate on $\mathbb{C} \setminus K(f)$, that is, $B_c^{-1}(e^{\mathbf{i}\theta} \cdot B_c)$. By means of straightening, one can define rotations on the complement of the filled Julia set of a polynomial-like map. It is not canonical as it depends on the choice of straightening map. However, its effect on the landing points of external rays is canonical.

Proposition 4.5. *Let $f : V_2 \to V_1$ be a polynomial-like map with connected filled Julia set $K(f)$. If $\phi : V_1 \setminus K(f) \to V_1 \setminus K(f)$ is a homeomorphism which commutes with f, then there exists a rotation of angle $2\pi j/(d-1)$, ρ_j, such that $\rho_j \circ \phi$ extends as the identity map onto $K(f)$.*

The proof is given in the Appendix.

Applying the above lemma to $\boldsymbol{\psi}_{n-1}^{-1} \circ g$ with $V_1 = Q_n^{\chi_n}$, $V_2 = P_n^{\chi_n}$, and an external ray connecting $\partial Q_n^{\chi_n}$ to $J_{n,0}$, we conclude that g can be extended as $\boldsymbol{\psi}_{n-1}$ onto $J_{n,0}$. It also follows from proof of the above lemma that g and $\boldsymbol{\psi}_{n-1}$ are homotopic relative $J_n \cup \partial Q_n^{\chi_n}$. That is because the quadrilaterals obtained in the above lemma cut the puzzle piece $Q_n^{\chi_n}$ into infinite number of topological disks such that g and $\boldsymbol{\psi}_{n-1}$ are equal on their boundaries.

Similarly, $\mathbf{h}_{n-1}$ can be adjusted on the other puzzle pieces $Q_{n,i}^{\chi_n}$, and moreover, $\mathbf{h}_{n-1}$ is homotopic to $\boldsymbol{\psi}_{n-1}$ on $Q_{n,i}^{\chi_n}$ relative $J_{n,i} \cup \partial Q_{n,i}^{\chi_n}$. We will denote the map obtained from extending $\mathbf{h}_{n-1}$ onto little Julia sets $J_{n,i}$ with the same notation $\mathbf{h}_{n-1}$.

Finally, we need to prepare $\mathbf{h}_{n-1}$ for the next step of the process. It is stated in the following lemma.

Lemma 4.6. *The K_2-q.c. map $\mathbf{h}_{n-1}$ can be adjusted (through a homotopy) on a neighborhood of $\cup_i J_{n,i}$ to a q.c. map $\mathbf{h}'_{n-1}$ which maps $\mathbf{V}_{n,i} := S_{n-1}(V_{n,i})$ onto $\widetilde{\mathbf{V}}_{n,i} := \widetilde{S}_{n-1}(\widetilde{V}_{n,i})$. Moreover, dilatation of $\mathbf{h}'_{n-1}$ is uniformly bounded by a constant $K_3(\varepsilon)$ depending only on ε.*

Proof. The basic idea is to continuously move all of $\mathbf{h}_{n-1}(\mathbf{V}_{n,i})$ (simultaneously) close enough to little Julia sets $J_{n,i}$, and then move them back to $\widetilde{\mathbf{V}}_{n,i}$. We will do this more precisely below. Let $\mathbf{U}_{n,i}$ denote $S_{n-1}(U_{n,i})$ and $\widetilde{\mathbf{U}}_{n,i}$ denote $\widetilde{S}_{n-1}(\widetilde{U}_{n,i})$.

The annuli $\mathbf{h}_{n-1}(\mathbf{V}_{n,i}) \setminus \widetilde{J}_{n,i}$ and $\widetilde{\mathbf{V}}_{n,i} \setminus \widetilde{J}_{n,i}$ have moduli bigger that ε/dK_1K_2 where K_2 is the dilatation of $\mathbf{h}_{n-1}$ and K_1 is the dilatation of S_{n-1}. Therefore, there exist topological disks $\widetilde{L}_{n,i} \supseteq \widetilde{J}_{n,i}$ with smooth boundaries and a constant $r > 0$ satisfying the following properties

- $\widetilde{L}_{n,i} \subset \mathbf{h}_{n-1}(\mathbf{V}_{n,i}) \cap \widetilde{\mathbf{V}}_{n,i}$,
- $\operatorname{mod}(\widetilde{L}_{n,i} \setminus \widetilde{J}_{n,i}) \geq r$,
- $\operatorname{mod}(\widetilde{\mathbf{V}}_{n,i} \setminus \widetilde{L}_{n,i}) \geq \varepsilon/2dK_1K_2$,
- $\operatorname{mod}(\mathbf{h}_{n-1}(\mathbf{V}_{n,i}) \setminus \widetilde{L}_{n,i}) \geq \varepsilon/2dK_1K_2$.

Now we claim that there exist q.c. maps

$$\chi_i : \big(\mathbf{h}_{n-1}(\mathbf{U}_{n,i}), \mathbf{h}_{n-1}(\mathbf{V}_{n,i}), \widetilde{L}_{n,i}, J_{n,i}\big) \to \big(D_5, D_3, D_2, D_1\big),$$

with uniformly bounded dilatation. That is because all the annuli $\widetilde{L}_{n,i} \setminus \widetilde{J}_{n,i}$, $\mathbf{h}_{n-1}(\mathbf{V}_{n,i}) \setminus \widetilde{L}_{n,i}$, and $\mathbf{h}_{n-1}(\mathbf{U}_{n,i}) \setminus \mathbf{h}_{n-1}(\mathbf{V}_{n,i})$ have moduli uniformly bounded from below and above independent of n and i.

The homotopy g_t: $\operatorname{Dom}(\mathbf{h}_{n-1}) \to \mathbb{C}$, for $t \in [0,1]$, is defined as

$$\begin{cases} \mathbf{h}_{n-1}(z) & \text{if } z \notin \bigcup_i \mathbf{V}_{n,i} \\ \chi_i^{-1}\Big(\big(\frac{-t}{3} \sin \frac{(|\chi_i \circ \mathbf{h}_{n-1}(z)|-1)\pi}{4} + 1\big) \cdot \chi_i \circ \mathbf{h}_{n-1}(z)\Big) & \text{if } z \in \mathbf{V}_{n,i}. \end{cases}$$

It is straight to see that $g_0 \equiv \mathbf{h}_{n-1}$ on $\operatorname{Dom} \mathbf{h}_{n-1}$, g_t is a well defined homeomorphism for every fixed t, and depends continuously on t for every fixed z. For every $z \in \partial \mathbf{V}_{n,i}$, at time $t = 1$, we have $g_1(z) = \chi_i^{-1}(\frac{2}{3} \cdot \chi_i \circ \mathbf{h}_{n-1}(z)) \in \partial \widetilde{L}_{n,i}$. That is, g_1 maps $\partial V_{n,i}$ to $\partial \widetilde{L}_{n,i}$. For the returning part, we consider a q.c. map

$$\Theta_i : \big(\widetilde{\mathbf{U}}_{n,i}, \widetilde{\mathbf{V}}_{n,i}, \widetilde{L}_{n,i}, \widetilde{J}_{n,i}\big) \to \big(D_5, D_3, D_2, D_1\big),$$

and define $g_{t+1} : \operatorname{Dom} \mathbf{h}_{n-1} \to \mathbb{C}$, for $t \in [0,1]$, as

$$\begin{cases} g_1(z) & \text{if } z \notin g_1^{-1}(\bigcup_i \widetilde{\mathbf{U}}_{n,i}) \\ \Theta_i^{-1}\Big(\big(\frac{t}{\sqrt{2}} \sin \frac{(|\chi_i \circ g_1(z)|-1)\pi}{4} + 1\big) \cdot \Theta_i \circ g_1(z)\Big) & \text{if } z \in g_1^{-1}(\widetilde{\mathbf{U}}_{n,i}). \end{cases}$$

The homotopy g_t for $t \in [0,2]$ is the desired adjustment. The map $g_2 : \operatorname{Dom} \mathbf{h}_{n-1} \to \operatorname{Range} \mathbf{h}_{n-1}$, is denoted by $\mathbf{h}'_{n-1}$. $\square$

Let $\Delta_{n-1,0}$ denote the S_{n-1}-preimage of the domain bounded by the equipotential of level $\eta(\varepsilon)$ in the dynamic plane of $\mathbf{f}_{c_{n-1}}$. The multiply connected domain $\Omega_{n-1,0}$ is defined as

$$\Delta_{n-1,0} \setminus \bigcup_{i=0}^{t_n/t_{n-1}} V_{n,it_{n-1}}.$$

The domains $\Delta_{n-1,i}$ and $\Omega_{n-1,i}$, for $i = 1, 2, \dots, t_{n-1}$, are defined as the pull back of $\Delta_{n-1,0}$ and $\Omega_{n-1,i}$, respectively, under f^{-i} along the orbit of the critical point.

Consider the map

$$h_{n-1,0} := \widetilde{S}_{n-1}^{-1} \circ \mathbf{h}'_{n-1} \circ S_{n-1} : \Delta_{n-1,0} \to \widetilde{\Delta}_{n-1,0}, \tag{4.3}$$

and then,

$$h_{n-1,i} := \tilde{f}^{-i} \circ h_{n-1,0} \circ f^i : \Delta_{n-1,i} \to \widetilde{\Delta}_{n-1,i}. \tag{4.4}$$

As these maps are compositions of two $K_1(\varepsilon)$-q.c., a $K_3(\varepsilon)$-q.c., and possibly some conformal maps, they are q.c. with a uniform bound on their dilatation. By our adjustment in Lemma 4.6 we have $h_{n-1,i}$ maps $\Omega_{n-1,i}$ onto $\widetilde{\Omega}_{n-1,i}$.

Finally, the annulus $V_{n-1,0} \setminus W_{n-1,0}$, with modulus bigger than ε/d, encloses $\Omega_{n-1,0}$ and is contained in $V_{n-1,0}$. This proves that the domain $\Omega_{n-1,0}$ is well inside the disk $D_{n-1,0} := V_{n-1,0}$. Similarly, appropriate preimages of $V_{n-1,0} \setminus W_{n-1,0}$ under the conformal maps f^{-i} introduce definite annuli around $\Omega_{n-1,i}$ which are contained in $D_{n-1,i} := V_{n-1,i}$. In this case, the topological disks $D_{n,i}$ are defined as the domains $V_{n,i}$ which contain the little Julia sets $J_{n,i}$ well inside themselves.

Case $\mathscr{B}$: Here, $\mathbf{f}_{c_{n-1}}$ is satellite renormalizable and its second renormalization is of primitive type. Let α_{n-1} denote the dividing fixed point of $\mathbf{f}_{c_{n-1}}$, and $\alpha_n \in J_{n-1,0}$ the dividing fixed point of it's first renormalization $\mathcal{R}\mathbf{f}_{c_{n-1}}$. By definition, the little Julia sets $J_{n,i}$ of $\mathbf{f}_{c_{n-1}}$ touch at the α_{n-1} fixed point. In this situation, the little Julia set of the primitive renormalization $\mathcal{R}\mathbf{f}_{c_{n-1}}$, and its forward images under $\mathbf{f}_{c_{n-1}}$ can be arbitrarily close to α_{n-1} (which is a non-dividing fixed point of $\mathcal{R}\mathbf{f}_{c_{n-1}}$). Our idea is to skip the satellite renormalization and start with the primitive one. This essentially imposes the secondary limbs condition on us.

Consider an equipotential of level $\eta(\varepsilon)$ contained in $S_{n-1}(W_{n-1,0})$, the external rays landing at α_{n-1}, and the external rays landing at the $\mathbf{f}_{c_{n-1}}$-orbit of α_n (see Figure 6). These rays and the equipotential $E^{\eta(\varepsilon)}$ depend holomorphically on the parameter within the secondary wake

$W(\eta)$ containing the parameter c_{n-1}. Let us denote $\mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}}$ by g for simplicity of notation throughout this case.

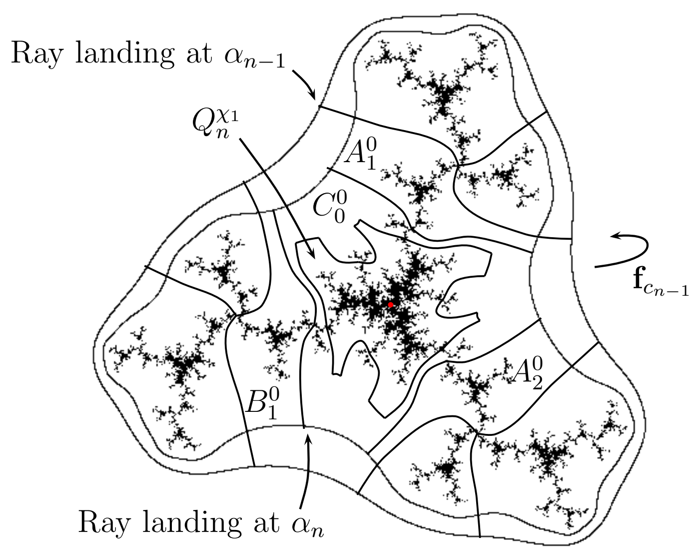

FIGURE 6. Figure of an infinitely renormalizable Julia set. The first renormalization is of satellite type and the second one is of primitive type. The puzzle piece $Q_n^{\chi_1}$ at the center is the first puzzle piece in the favorite nest.

Now, using the above rays and equipotential, we introduce some new puzzle pieces. Let Y_0^0, as before, denote the puzzle piece containing the critical point which is bounded by $E(\eta)$, the external rays landing at α_{n-1}, and the external rays landing at $\mathbf{f}_{c_{n-1}}$-preimage of α_{n-1} (i.e., at all $\omega\alpha_{n-1}$ with ω a dth root of unity). The external rays landing at α_n and their g-preimage, cut the puzzle piece Y_0^0 into finitely many pieces. Let us denote the one containing the critical point by C_0^0, the non-critical ones which have a boundary ray landing at α_n by B_i^0, and the rest of them by A_j^0 (these ones have a boundary external ray landing at some $\omega\alpha_n$).

The g-preimage of Y_0^0 along the postcritical set is contained in itself. As all processes of making modified principal nest and the pseudo-conjugacy in Theorem 3.2 are based on pullback arguments, the same ideas are applicable here. The only difference is that we do not have equipotentials for the second renormalization. As we will see in a moment, certain external rays and part of the equipotential bounding Y_0^0

will play the role of an equipotential for the second renormalization of $\mathbf{f}_{c_{n-1}}$.

By definition of satellite and primitive renormalizability, $g^n(0)$ belongs to Y_0^0, for $n \geq 0$, and there is a first moment t with $g^t(0) \in A_1^0$ (by rearranging the indices if required). Pulling back A_1^0 under g^t along the critical orbit, we obtain a puzzle piece $Q_n^{\chi_1} \ni 0$, such that $C_0^0 \setminus Q_n^{\chi_1}$ is a non-degenerate annulus. That is because C_0^0 is bounded by the external rays landing at α_n and their g-preimage. Therefore, if $Q_n^{\chi_1}$ intersects ∂C_0^0 at some point on the rays, orbit of this intersection under g^k, for $k \geq 1$, stays on the rays landing at α_n. This implies that image of $Q_n^{\chi_1}$ can never be A_1^0. Also, they do not intersect at equipotentials, as they have different levels.

Now, let $m > t$ be the smallest integer with $g^m(0) \in Q_n^{\chi_1}$. Pulling back $Q_n^{\chi_1}$ under g^m along the critical orbit we obtain $P_n^{\chi_1}$. The map g^m is a unicritical degree d branched covering from $P_n^{\chi_1}$ onto $Q_n^{\chi_1}$. This introduces the first two pieces in the favorite nest. The rest of the process to form the whole favorite nest is the same as in Section 3.2.

Consider the map $\mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}} : Y_0^0 \to \mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}}(Y_0^0)$, and the corresponding tilde one. One applies Theorem 3.2 to these maps, using the favorite nests introduced in the above paragraph, to obtain a q.c. pseudo-conjugacy

$$\mathbf{h}_{n-1} : \mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}}(Y_0^0) \to \widetilde{\mathbf{f}}_{c_{n-1}}^{t_n/t_{n-1}}(\widetilde{Y}_0^0),$$

up to level of $Q_n^{\chi_n}$. The equipotential of level $\eta(\varepsilon)$, the external rays landing at α_{n-1}, and the external rays landing at the $\mathbf{f}_{c_{n-1}}$-orbit of α_n depend holomorphically on the parameter within the secondary wake $W(\eta)$ containing the parameter c_{n-1}. Therefore, by Proposition 3.3, the dilatation of $\mathbf{h}_{n-1}$ depends on the hyperbolic distance between c_{n-1} and $\tilde{c}_{n-1}$ within one of the finite secondary wakes $W(\eta)$ under our combinatorial assumption. Like previous case, modulus of the top annulus $Q_n^{\chi_1} \setminus P_n^{\chi_1}$ is bounded below for parameters in these secondary limbs. Thus, the quasi-conformal dilatation depends on the *a priori* bounds ε and the combinatorial class $\mathcal{SL}$.

As $\mathbf{f}_{c_{n-1}}^j : \mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}-j}(Y_0^0) \to \mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}}(Y_0^0)$, for $j = 1, 2, \dots, t_n/t_{n-1} - 1$, is univalent, we can lift $\mathbf{h}_{n-1}$ on other puzzle pieces as

$$\mathbf{h}_{n-1} := \widetilde{\mathbf{f}}_{c_{n-1}}^{-j} \circ \mathbf{h}_{n-1} \circ \mathbf{f}_{c_{n-1}}^j : \mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}-j}(Y_0^0) \to \widetilde{\mathbf{f}}_{c_{n-1}}^{t_n/t_{n-1}-j}(\widetilde{Y}_0^0)$$

for these j's. Since all these maps match the Böttcher marking, they fit together to build a q.c. map from a neighborhood of $J(\mathbf{f}_{c_{n-1}})$ to a neighborhood of $J(\widetilde{\mathbf{f}}_{c_{n-1}})$. Further, it can be extended as the identity map in the Böttcher coordinates to a q.c. map from the domain bounded by equipotential $E^{\eta(\varepsilon)}$ to the corresponding tilde domain.

Finally, by the argument after Proposition 4.5 and Lemma 4.6, we adjust $\mathbf{h}_{n-1}$ to obtain a q.c. map $\mathbf{h}'_{n-1}$ that satisfies

$$\mathbf{h}'_{n-1}(S_{n-1}(V_{n+1,it_{n-1}}))=\widetilde{S}_{n-1}(\widetilde{V}_{n,it_{n-1}}),$$

and

$$\mathbf{h}'_{n-1}(J(\mathcal{R}\mathbf{f}_{c_{n-1}})) = J(\mathcal{R}\widetilde{\mathbf{f}}_{c_{n-1}})$$

for $i = 0, 1, 2, \ldots, t_{n+1}/t_{n-1} - 1$.

Now, $\Delta_{n-1,0}$ is defined as S_{n-1}-pullback of the domain bounded by the equipotential $E^{\eta(\varepsilon)}$. The domain $\Omega_{n-1,0}$ is

$$\Delta_{n-1,0} \setminus \bigcup_{i=0}^{t_{n+1}/t_{n-1}-1} V_{n+1,it_{n-1}}.$$

The regions $\Delta_{n-1,i}$ and $\Omega_{n-1,i}$, for $i = 1, 2, 3, \ldots, t_{n-1}$, are pullbacks of $\Delta_{n-1,0}$ and $\Omega_{n-1,0}$ under f^i, respectively. Like previous case, $h_{n-1,i}$ is defined as in Equations 4.3 or 4.4 and satisfies

$$h_{n-1,i}(\Omega_{n-1,i}) = \widetilde{\Omega}_{n-1,i}, \text{ for } i = 1, 2, \ldots, t_{n-1}.$$

For the same reason as in Case $\mathscr{A}$, $\Omega_{n-1,i}$ is well inside $D_{n-1,i} := V_{n-1,i}$.

Case $\mathscr{C}$: The argument in this case relies on the compactness of the parameters under consideration rather than a dynamical discussion.

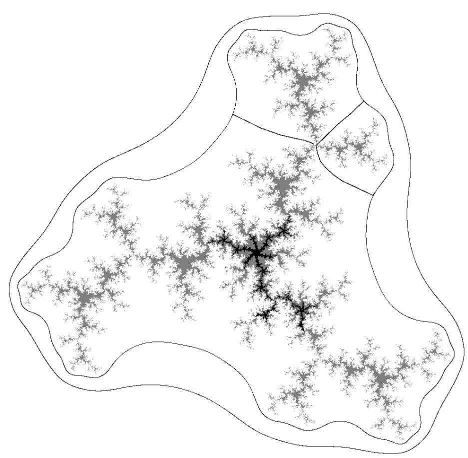

FIGURE 7. A twice satellite renormalizable Julia set drawn in grey. The dark part is the Julia Bouquet $\mathbf{B}_{2,0}$.

Little Julia sets $\mathbf{J}_{1,i}$ of the first renormalization of $\mathbf{f}_{c_{n-1}}$ touch at the dividing fixed point α_{n-1} of $\mathbf{f}_{c_{n-1}}$. Note that α_{n-1} is one of the non-dividing fixed points of the first renormalization of $\mathbf{f}_{c_{n-1}}$. The union of these little Julia sets is called *Julia bouquet* and denoted by $\mathbf{B}_{1,0}$. Similarly, the little Julia sets $\mathbf{J}_{2,i}$, for $i = 0, 1, 2, \ldots, (t_{n+1}/t_{n-1}) - 1$, of the second renormalization of $\mathbf{f}_{c_{n-1}}$ are organized in pairwise disjoint *bouquets* $\mathbf{B}_{2,j}$. That is, each $\mathbf{B}_{2,j}$ consists of t_{n+1}/t_n little Julia sets $\mathbf{J}_{2,i}$ touching at one of their non-dividing fixed points. As usual, $\mathbf{B}_{2,0}$ denotes the bouquet containing the critical point. See Figure 7.

By an equipotential of level $\eta(\varepsilon)$, contained in $S_{n-1}(W_{n-1,0})$, and the external rays landing at α_{n-1}, we form the puzzle pieces of level zero. Recall that Y_0^0 denotes the one containing the critical point. The following lemma states that the bouquets are well apart form each other.

Lemma 4.7. *For all parameters in a finite number of truncated secondary limbs, modulus of the annulus $Y_0^0 \setminus \mathbf{B}_{2,0}$ is uniformly bounded above and below.*

Let X and Y be compact subsets of $\mathbb{C}$ equipped with the Euclidean metric d. The *Hausdorff distance* between X and Y is defined as

$$d_H(X, Y) := \inf\{\varepsilon \in \mathbb{R} : Y \subset B_\varepsilon(X), \text{ and } X \subset B_\varepsilon(Y)\}.$$

The space of all compact subsets of $\mathbb{C}$ endowed with this metric is a complete metric space.

A set valued map $c \mapsto X_c$, with X_c compact in $\mathbb{C}$, is called *upper semi-continuous* if $c_n \to c$ implies that X_c contains Hausdorff limit of any convergent subsequence of X_{c_n}.

We say a family of simply connected domains U_λ, parametrized on a disk, depends continuously on λ, if there exit choices of uniformizations $\psi_\lambda : D_1 \to U_\lambda$ continuous in both variables. We say a family of polynomial-like maps $(P_\lambda : V_\lambda \to U_\lambda, \lambda \in \Lambda)$, parametrized on a topological disk Λ, depends continuously on λ, if U_λ is a continuous family of simply connected domains in $\mathbb{C}$, and for every fixed $z \in \mathbb{C}$, $P_\lambda(z)$ depends continuously on λ wherever it is defined.

Proposition 4.8. *Let $(P_\lambda : V_\lambda \to U_\lambda, \lambda \in \Lambda)$ be a continuous family of polynomial like maps with connected filled Julia sets K_λ. The map $\lambda \mapsto K_\lambda$ is upper semi-continuous.*

Proof. Assume that $\lambda_n \to \lambda$, $K_n := K(P_{\lambda_n})$, and $K_\lambda := K(P_\lambda)$. To prove that limit of every convergent subsequence of K_n is contained in K_λ, it is enough to show that for every $\varepsilon > 0$, $K_n \subseteq B_\varepsilon(K_\lambda)$ for sufficiently large n.

To see this, first assume that $z \notin B_\varepsilon(K_\lambda)$. If $z \notin V_\lambda$, then by continuous dependence of V_λ on λ, $z \notin K_n$ for large n. If $z \in V_\lambda$, then there exists a positive integer l with $P_\lambda^l(z) \in U_\lambda \setminus V_\lambda$. As $P_{\lambda_n} : V_{\lambda_n} \to U_{\lambda_n}$ converges to $P_\lambda : V_\lambda \to U_\lambda$, $P_{\lambda_n}^l(z)$ or $P_{\lambda_n}^{l+1}(z)$ belongs to $U_{\lambda_n} \setminus V_{\lambda_n}$. Therefore, $z \notin K_n$. □

Proof of Lemma 4.7. Let c_{n-1} be a twice satellite renormalizable parameter in a given truncated secondary limb. Consider the external rays landing at the dividing fixed point α_n of $\mathcal{R}\mathbf{f}_{c_{n-1}}$ and their preimage under $\mathcal{R}\mathbf{f}_{c_{n-1}}$. Let X_0^0 denote the critical puzzle piece obtained from these rays. As $\mathbf{f}_{c_{n-1}}$ is twice satellite renormalizable, $\mathcal{R}^2\mathbf{f}_{c_{n-1}} : X_0^0 \to \mathbb{C}$ is a branched covering over its image. One can consider a continuous thickening of X_0^0, described in Section 3.2, to form a continuous family of polynomial-like maps parametrized over this truncated limb. The little Julia set of this map is $\mathbf{J}_{2,0}$, and the Julia bouquet $\mathbf{B}_{2,0}$ is the connected component of

$$\bigcup_{i=0}^{t_{n+1}/t_{n-1}-1} \mathbf{f}_{c_{n-1}}(\mathbf{J}_{2,0})$$

containing the critical point.

For c_{n-1} in a finite number of truncated secondary limbs, the Julia bouquet $\mathbf{B}_{2,0}$ is union of a finite number of little Julia sets. Hence, by above lemma, it depends upper semi-continuously on c_{n-1}. As $\mathbf{B}_{2,0}$ is contained well inside the interior of Y_0^0 for c_{n-1} in the closure of the truncated secondary limb, we conclude that modulus of $Y_0^0 \setminus \mathbf{B}_{2,j}$ is uniformly bounded below.

To see that these moduli are uniformly bounded above, one only needs to observe that α_n and 0 belong to $\mathbf{B}_{2,0}$ and are distinct for these parameters. □

By Lemma 4.7 there are simply connected domains $\mathbf{L}'_n \subseteq \mathbf{L}_n$ and $\widetilde{\mathbf{L}}'_n \subseteq \widetilde{\mathbf{L}}_n$ such that moduli of the annuli

$$\begin{array}{lll} Y_0^0 \setminus \mathbf{L}_n, & \mathbf{L}_n \setminus \mathbf{L}'_n, & \mathbf{L}'_n \setminus \mathbf{B}_{2,0} \\ \widetilde{Y}_0^0 \setminus \widetilde{\mathbf{L}}_n, & \widetilde{\mathbf{L}}_n \setminus \widetilde{\mathbf{L}}'_n, & \widetilde{\mathbf{L}}'_n \setminus \widetilde{\mathbf{B}}_{2,0} \end{array} \tag{4.5}$$

are bounded above and below by some constants depending only on the combinatorial class $\mathcal{SL}$. It follows that, the ratios

$$\frac{\operatorname{mod}(Y_0^0 \setminus \mathbf{L}_n)}{\operatorname{mod}(\widetilde{Y}_0^0 \setminus \widetilde{\mathbf{L}}_n)}, \quad \frac{\operatorname{mod}(\mathbf{L}_n \setminus \mathbf{L}'_n)}{\operatorname{mod}(\widetilde{\mathbf{L}}_n \setminus \widetilde{\mathbf{L}}'_n)}, \quad \frac{\operatorname{mod}(\mathbf{L}'_n \setminus \mathbf{B}_{2,0})}{\operatorname{mod}(\widetilde{\mathbf{L}}'_n \setminus \widetilde{\mathbf{B}}_{2,0})}$$

are also uniformly bounded below and above independent of n.

Above data implies that there exits a q.c. map

$$\mathbf{h}_{n-1} : Y_0^0 \setminus \mathbf{L}_n \to \widetilde{Y}_0^0 \setminus \widetilde{\mathbf{L}}_n,$$

with a uniform bound on its dilatation, which matches the Böttcher marking on the boundary of Y_0^0 (See Lemma 4.11). We further lift $\mathbf{h}_{n-1}$ via $\mathbf{f}_{c_{n-1}}^{-i}$ and $\mathbf{f}_{\tilde{c}_{n-1}}^{-i}$ to extend $\mathbf{h}_{n-1}$ to q.c. maps

$$\mathbf{h}_{n-1} : \mathbf{f}_{c_{n-1}}^{-i}(Y_0^0 \setminus \mathbf{L}_n) \to \tilde{\mathbf{f}}_{c_{n-1}}^{-i}(\widetilde{Y}_0^0 \setminus \widetilde{\mathbf{L}}_n), \text{ for } i = 1, 2, \ldots, \frac{t_n}{t_{n-1}} - 1$$

with the same dilatation. The domain of each such map is a puzzle piece Y_j^0 (with $j = t_n/t_{n-1} - i$) cut off by the equipotential of level η/d^l and a component of $\mathbf{f}_{c_{n-1}}^{-i}(\partial \mathbf{L}_n)$. As all these maps match the Böttcher marking on the boundaries of Y_j^0, they can be glued together. Finally, one extends this map as the identity in the Böttcher coordinates onto spaces between equipotential of level η and equipotentials of level η/d^l. we denote this extended map with the same notation $\mathbf{h}_{n-1}$. As it is lifted under and extended by holomorphic maps, there is a uniform bound on its dilatation.

Let $\Delta_{n-1,0}$ be the S_{n-1}-preimage of the domain inside equipotential $E(\eta)$, and $L_{n,i}$ be the component of the S_{n-1}-preimage of $\mathbf{f}_{c_{n-1}}^{-i}(\mathbf{L}_n)$ enclosing the little post critical set $J_{n-1,i} \cap \mathcal{PC}(f)$. We define the multiply connected regions

$$\Omega_{n-1,0} := \Delta_{n-1,0} \setminus \bigcup_{i=0}^{t_n/t_{n-1}-1} L_{n,it_{n-1}}.$$

Like before, $\Delta_{n-1,i}$ and $\Omega_{n-1,i}$ are defined as f^i-preimage of $\Delta_{n-1,0}$ and $\Omega_{n-1,0}$ containing or enclosing $J_{n-1,i}$, respectively.

We have

$$h_{n-1,0} := \widetilde{S}_{n-1}^{-1} \circ \mathbf{h}_{n-1} \circ S_{n-1} : \Delta_{n-1,0} \to \widetilde{\Delta}_{n-1,0},$$

$$\text{with } h_{n-1,0}(\Omega_{n-1,0}) = \widetilde{\Omega}_{n-1,0}.$$

Also,

$$h_{n-1,i} := \tilde{f}^{-i} \circ h_{n-1,0} \circ f^i : \Delta_{n-1,i} \to \widetilde{\Delta}_{n-1,i},$$

$$\text{with } h_{n-1,i}(\Omega_{n-1,i}) = \widetilde{\Omega}_{n-1,i}.$$

As the equipotential $\eta(\varepsilon)$ is contained in $S_{n-1}(W_{n-1,0})$, $\Omega_{n-1,0}$ is contained in $W_{n-1,0}$. Therefore, $\Omega_{n-1,0}$ is well inside $D_{n-1,0} := V_{n-1,0}$. Conformal invariance of modulus implies that the other domains $\Omega_{n-1,i}$ are also well inside $D_{n-1,0} := V_{n-1,i}$. This completes the construction in Case $\mathscr{C}$.

To fit together the multiply connected domains $\Omega_{n(k),i}$ and the q.c. maps $h_{n(k),i} : \Omega_{n(k),i} \to \widetilde{\Omega}_{n(k),i}$, we follow the word of cases introduced at the beginning of the construction. In Cases $\mathscr{A}$ and $\mathscr{B}$, we have adjusted $\mathbf{h}_{n-1}$, see Lemma 4.6, such that it sends $\partial V_{n,0}$ to $\partial \widetilde{V}_{n,0}$. Therefore, if any of the three cases follows Case $\mathscr{A}$ or $\mathscr{B}$, we consider $\mathcal{R}^n f : W_{n,0} \to V_{n,0}$ and straighten it with these choices of domains (instead of $\mathcal{R}^n f : V_{n,0} \to U_{n,0}$). If a case of construction on level n follows Case $\mathscr{C}$, the set $\mathbf{\Delta}_{n,0}$ introduced on level n is not contained in the hole $D_{n,0} = L_{n,0}$ of the previous level. The following paragraph explains a similar adjustment needed here.

As the annulus $\mathbf{\Delta}_{n,0} \setminus J(\mathbf{f}_{c_n})$ has a definite modulus in terms of ε, the annulus $\mathbf{\Delta}_{n,0} \setminus \mathbf{B}_{1,0}$, where $\mathbf{B}_{1,0}$ is the only Julia bouquet of $\mathbf{f}_{c_{n-1}}$, also has a definite modulus in terms of ε. By quasi-invariance of modulus, the annulus

$$S_n(L'_{n,0} \cap \text{Dom } S_n) \setminus \mathbf{B}_{1,0}$$

also has a definite modulus in terms of ε. Now let $\mathbf{E}_n$ (and corresponding tilde one) be a topological disk contained in

$$S_n(L'_{n,0} \cap \text{Dom } S_n) \cap \mathbf{\Delta}_{n,0}$$

with mod $(E_n \setminus \mathbf{B}_{1,0})$ bigger than a constant in terms of ε. By a similar argument as in Lemma 4.6 we adjust $\mathbf{h}_n$ through a homotopy to obtain a map $\mathbf{h}'_n : \mathbf{E}_n \to \widetilde{\mathbf{E}}_n$. Now, in this situation $\mathbf{\Delta}_{n,0}$ is replaced by $\mathbf{E}_n$, $\Delta_{n,0}$ by $S_n^{-1}(\mathbf{E}_n)$, and h_n by $\widetilde{S}_n \circ \mathbf{h}'_n \circ S_n$. The annulus $L_n \setminus L'_n$ provides a definite space separating $\Omega_{n,0}$ and $L_{n,0}$.

In the following two sections, we will denote the holes of $\Omega_{n,i}$ by $\mathbb{V}_{n+1,j}$, that is, $\mathbb{V}_{n+1,j}$ is $V_{n+1,j}$, if n belongs to Case $\mathscr{A}$ or $\mathscr{B}$, and $\mathbb{V}_{n+1,j}$ is $S_n^{-1}(\mathbf{L}_{n,j})$ if n belongs to Case $\mathscr{C}$.

4.4. **The gluing maps $g_{n(k),i}$.** In this section we build $K'(\varepsilon)$-q.c. maps

$$g_{n(k),i} : \mathbb{V}_{n(k),i} \setminus \Delta_{n(k),i} \to \widetilde{\mathbb{V}}_{n(k),i} \setminus \widetilde{\Delta}_{n(k),i}.$$

Every $g_{n(k),i}$ must be identical with $h_{n(k-1),i}$ on $\partial \mathbb{V}_{n(k),i}$, and with $h_{n(k),i}$ on $\partial \Delta_{n(k),i}$ (which is outer boundary of $\Omega_{n(k),i}$). Then gluing all these maps $g_{n(k),i}$ and $h_{n(k),i}$ together produces a q.c. map H with dilatation bounded by maximum of K and $K'(\varepsilon)$. In what follows, for simplicity of notation, we use index n instead of $n(k)$, and assume that n runs over the subsequence $n(k)$. So for all n, means for all $n(k)$'s.

Like previous steps, it is enough to build $g_{n,0}$ for all n, and lift them via f^{-i} and $\tilde{f}^{-i}$ to obtain $g_{n,i}$, for $i = 1, 2, \ldots, t_n$. Definition of the maps $h_{n,i}$ as well as the domains $\mathbb{V}_{n,i}$ and $\Omega_{n,i}$ implies that these maps also glue together on the boundaries of their domains of definition.

Again, we drop the second subscript index if it is 0, i.e., g_n denotes the map $g_{n,0}$.

To build a q.c. map from an annulus to another annulus with given boundary conditions, there is a choice of the number of "twists" one may make. To have a uniform bound on the dilatation of such a map, not only the two annuli must have proportional moduli uniformly bounded below, the number of twists must be uniformly bounded as well. Note that these twists change the homotopy class of the final map H.

In this section, we show that the corresponding annuli $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$ and $\widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{\Delta}_{n,0}$ have proportional moduli (with a constant depending only on ε). In the next section we prescribe the correct number of twists needed to be in the isotopy class of a Thurston conjugacy.

Lemma 4.9. *There exists a constant M', depending only on ε, such that*

$$\frac{1}{M'} \leq \frac{\text{mod } (\widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{\Delta}_{n,0})}{\text{mod } (\mathbb{V}_{n,0} \setminus \Delta_{n,0})} \leq M'.$$

Proof. If level n follows one of the Cases $\mathscr{A}$ or $\mathscr{B}$, then

$$\text{mod } (\mathbb{V}_{n,0} \setminus \Delta_{n,0}) \leq \text{ mod } (V_{n,0} \setminus J_{n,0}) \leq 2\eta,$$

for some constant η, by our assumptions on page 19 on the annuli. If level n follows a Case $\mathscr{C}$ then

$$\text{mod } (\mathbb{V}_{n,0} \setminus \Delta_{n,0}) \leq \text{ mod } S_n^{-1}(\mathbf{L}_{n,0} \setminus \mathbf{B}_{1,0}) \leq M''$$

for some constant M'' by the statement in 4.5. Similarly, modulus of $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$ is bigger than ε or some constant depending on ε, whether level n follows a Case $\mathscr{A}$, $\mathscr{B}$, or $\mathscr{C}$. Hence, moduli of the annuli $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$ and $\widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{\Delta}_{n,0}$ are pinched between constants depending only on ε. This implies the lemma. $\square$

Assume that $A(r)$ denotes the round annulus $D_r \setminus D_1$, for $r > 1$, and let $\gamma : [0, 1] \to A(r)$ denote a curve parametrized in the polar coordinate as $\gamma(t) = (r(t), \theta(t))$, for $t \in [0, 1]$, with $r(t)$ and $\theta(t)$ continuous functions from $[0, 1]$ to $\mathbb{R}$. The *wrapping number* of γ, denoted by $\omega(\gamma)$, is defined as $\theta(1) - \theta(0)$. Given a curve $\gamma : [0, 1] \to U$, where U is an annulus, with $\gamma(0)$ on the inner boundary of U (corresponding to the bounded component of $\mathbb{C} \setminus U$) and $\gamma(1)$ on the outer boundary of U (corresponding to the unbounded one), we define the wrapping number of γ in U, as $\omega(\gamma) := \omega(\phi \circ \gamma)$, where ϕ is a uniformization of the annulus U by a round annulus. Note that $\omega(\gamma)$ is invariant under the automorphism group of U. So, it is independent of the choice of

uniformization. In addition, just like winding number, it is constant over the homotopy class of all curves with the same boundary points.

Proposition 4.10. *Given fixed constants $K \geq 1$ and $r > 1$, there exists a constant N such that for every K-q.c. map $\psi : A(r) \to A(r')$, the wrapping number of the curve $\psi(t), t \in [1, r]$, belongs to the interval $[-N, N]$.*

Proof. This follows from compactness of the class of K-q.c. maps from $A(r)$ to some $A(r')$. □

In the following lemma let θ be a branch of argument defined on $\mathbb{C}$ minus an straight ray going from 0 to infinity.

Lemma 4.11. *Fix round annuli $A(r)$, $A(r')$, positive constants δ and K_1, as well as an integer k with*

$$\bmod A(r')/K_1 \leq \bmod A(r) \leq K_1 \bmod A(r'), \text{ and } \bmod A(r) \geq \delta.$$

If homeomorphisms

$$h_1 : \partial D_r \to \partial D_{r'} \text{ and } h_2 : \partial D_1 \to \partial D_1$$

have K_2-q.c. extensions to some neighborhoods of these circles for some K_2, then there exists a K-q.c. map $h : A(r) \to A(r')$ such that

- *$h(z) = h_1(z)$ for every $z \in \partial D_r$, and $h(z) = h_2(z)$ for every $z \in \partial D_1$.*
- *The curve $h(t)$, for $t \in [1, r]$, has wrapping number*

$$\theta(h_1(r)) - \theta(h_2(1)) + 2k\pi.$$

Moreover, K depends only on K_1, K_2, k and δ.

If $h_1(r)$ or $h_2(1)$ does not belong to domain of θ, one may compose h_1 and h_2 with a small rotation. By adding a $+1$ or -1 to k, the statement still holds independent of the choice of this rotation. One proves this statement by explicitly building such maps for every k. Further details are given in the Appendix.

Applying the above lemma to the uniformization of $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$ and $\widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{\Delta}_{n,0}$, with the induced maps from $h_{n-1,0}$ and $h_{n,0}$ on their boundaries, and an integer k_n, which is determined later, gives the needed K'-q.c. maps g_n. In the next section we prescribe some special numbers k_n, which are bounded by a constant depending on ε, in order to make the K-q.c. map H, obtained after gluing all these maps, homotopic to a topological conjugacy relative the postcritical set.

Definite moduli of the annuli $\mathbb{V}_{n,i} \setminus \Delta_{n,i}$ implies that the holes $\mathbb{V}_{n,i}$ shrink to points in the postcritical set. Therefore, H can be extended to a well defined K-q.c. map on the postcritical set. See [Str55] for a

detailed proof of this statement and further results on quasi-conformal removability of sets.

4.5. **Isotopy.** Denote by $\boldsymbol{\psi}_n$ the topological conjugacy between $\mathbf{f}_{c_n}$ and $\mathbf{f}_{\tilde{c}_n}$ obtained from extending the identity map in the Böttcher coordinates onto Julia sets. The lift $\psi_{n,0} := \widetilde{S}_n^{-1} \circ \boldsymbol{\psi}_n \circ S_n$ topologically conjugates $\mathcal{R}^n f$ and $\mathcal{R}^n \tilde{f}$ on a neighborhood of the little Julia set $J_{n,0}$. Note that this neighborhood covers the domain $\Omega_{n,0}$. In the dynamic plane of $\mathbf{f}_{c_n}$, let $U(\eta)$ denote the domain enclosed by the equipotential of level η.

Lemma 4.12. *The q.c. map $h_{n,i} : \Delta_{n,i} \to \widetilde{\Delta}_{n,i}$, for $i = 0, 1, \dots t_n - 1$, is homotopic to*

$$\psi_{n,i} := f^{-i} \circ \psi_{n,0} \circ f^i : \Delta_{n,i} \to \mathbb{C}$$

relative the little Julia sets $J_{n+1,j}$ of level $n+1$ inside $\Delta_{n,i}$.

Note that $\psi_{n,i}(\Delta_{n,i})$ is a neighborhood of the little Julia sets $\widetilde{J}_{n+1,i+t_n j}$ contained in $\widetilde{\Delta}_{n,i}$.

Proof. By definition of the domains $\Delta_{n,i}$ and $\mathbb{V}_{n,i}$, as well as the q.c. maps $h_{n,i}$ it is enough to prove the statement for $i = 0$. For the other ones one lifts the homotopies via f^i and $\tilde{f}^{-i}$, or defines them in a similar manner.

As $\Delta_{n,0}$, $\psi_{n,0}$, and the q.c. map $h_{n,0}$ are lifts of $\boldsymbol{\Delta}_{n,0}$, $\boldsymbol{\psi}_n$, and $\mathbf{h}'_{n,0}$ under the straightening map, it is enough to make the homotopy on the dynamic planes of $\mathbf{f}_{c_n}$ and $\mathbf{f}_{\tilde{c}_n}$ and then transfer it to the dynamic planes of $\mathcal{R}^n f$ and $\mathcal{R}^n \tilde{f}$ by the straightening map. Recall that in our construction, $\mathbf{h}'_{n,0}$ is an adjustment of $\mathbf{h}_{n,0}$ through a homotopy relative the little Julia sets of level $n+1$. Thus, to prove the lemma, we only need to show that $\mathbf{h}_{n,0}$ and $\boldsymbol{\psi}_n$ are homotopic relative the little Julia sets.

First assume that the level n belongs to Case $\mathscr{A}$. The idea of the proof, presented below in detail, is to divide the domain $\boldsymbol{\Delta}_{n,0}$, by means of rays and equipotential arcs, into some topological disks and one annulus such that $\boldsymbol{\psi}_n$ and $\mathbf{h}_{n,0}$ are identical on the boundaries of these domains.

Recall the puzzle piece $Q_{n,0}^{\chi_n}$ (where $Q_{n,0}^{\chi_n} = Y_0^{q_{\chi_n}}$). The equipotential $\mathbf{f}_{c_n}^{-\chi_n}(E^\eta)$, and the rays bounding $Q_{n,i}^{\chi_n}$, for $i = 0, 1, \dots, t_n - 1$, up to equipotential $\mathbf{f}_{c_n}^{-\chi_n}(E^\eta)$, cut the domain $\boldsymbol{\Delta}_{n,0}$ into one annulus $\boldsymbol{\Delta}_{n,0} \setminus \mathbf{f}^{-\chi_n}(U(\eta))$ and some topological disks. The topological disks which do not intersect the little Julia sets of level n, the puzzle pieces $Q_{n,i}^{\chi_n}$, and the remaining annulus $U(\eta) \setminus f^{-\chi_n}(U(\eta))$ are the appropriate domains.

By Theorem 3.2, the maps $\mathbf{h}_{n,0}$ and $\boldsymbol{\psi}_n$ are identical in the Böttcher coordinate on the boundaries of these domains. Indeed, the topological conjugacy $\boldsymbol{\psi}_n$ between $\mathbf{f}_{c_n}$ and $\mathbf{f}_{\tilde{c}_n}$ is the identity map in the Böttcher coordinates, and the pseudo-conjugacy $h_{n,0}$ obtained in Theorem 3.2 also matches the Böttcher marking. This proves that the two maps are homotopic outside of the puzzle pieces $Q^{\chi_n}_{n,i}$ relative $\cup_i \partial Q^{\chi_n}_{n,i}$.

To define a homotopy inside $Q^{\chi_n}_{n,0}$, recall that we started with a q.c. map g from $Q^{\chi_n}_{n,0} \setminus P^{\chi_n}_{n,0}$ to the coresponding tilde one, which was homotopic to $\boldsymbol{\psi}_n$ relative $\partial(Q^{\chi_n}_{n,0} \setminus P^{\chi_n}_{n,0})$. Hence, all lifts of g from the annuli $A^k_n \setminus A^{k+1}_n$ to the corresponding tilde ones, considered in Case $\mathscr{A}$, are homotopic to $\boldsymbol{\psi}_n$ relative $\partial(A^k_n \setminus A^{k+1}_n)$. As the two maps are identical on the little Julia set $J_{n+1,0}$ inside $Q^{\chi_n}_{n,0}$, this defines a global homotopy between g and $\boldsymbol{\psi}_n$ on $Q^{\chi_n}_{n,0}$ relative $\partial Q^{\chi_n}_{n,0} \cup J_{n+1,0}$.

Above argument applies to all other domains $Q^{\chi_n}_{n,i}$ as well.

If level n belongs to Case $\mathscr{B}$, we repeat the above argument on each puzzle pieces of level zero. If level n follows a Case $\mathscr{C}$, one considers the same homotopies but restricted to a smaller set. $\square$

Assume level n belongs to Case $\mathscr{A}$ or $\mathscr{B}$, and it follows a Case $\mathscr{A}$ or $\mathscr{B}$. Consider uniformizations

$$\phi_1 : A(s) \to (\mathbb{V}_{n,0} \setminus K_{n,0}), \quad \phi_2 : A(r) \to (\Delta_{n,0} \setminus K_{n,0})$$
$$\widetilde{\phi}_1 : A(\tilde{s}) \to (\widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{K}_{n,0}), \quad \widetilde{\phi}_2 : A(\tilde{r}) \to (\widetilde{\Delta}_{n,0} \setminus \widetilde{K}_{n,0})$$

by round annuli. The q.c. maps $h_{n-1,0} : \mathbb{V}_{n,0} \setminus K_{n,0} \to \widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{K}_{n,0}$ and $h_{n,0} : \Delta_{n,0} \setminus K_{n,0} \to \widetilde{\Delta}_{n,0} \setminus \widetilde{K}_{n,0}$ lift via ϕ_i and $\widetilde{\phi}_i$, $i = 1, 2$, to q.c. maps $\hat{h}_{n-1,0} : A(s) \to A(\tilde{s})$ and $\hat{h}_{n,0} : A(r) \to A(\tilde{r})$ with the same dilatation. By composing these uniformizations with rotations, if necessary, we may assume that the point one is mapped to the point one under $\hat{h}_{n-1,0}$ and $\hat{h}_{n,0}$. By Proposition 4.10, image of the line segments $[1, s]$ and $[1, r]$ under the q.c. maps $\hat{h}_{n-1}$ and $\hat{h}_{n,0}$, respectively, have wrapping numbers $\omega_{1,n}$ and $\omega_{2,n}$ whose absolute values are bounded by some constant depending only on ε. Define k_n as $\omega_{1,n} - \omega_{2,n}$. Let $g'_n : D_s \setminus D_r \to D_{\tilde{s}} \setminus D_{\tilde{r}}$ be a gluing of $\hat{h}_{n,0} : D_r \to D_{\tilde{r}}$ and $\hat{h}_{n-1,0} : D_s \to D_{\tilde{s}}$, using Lemma 4.11, with k_n number of twists. With this choice of gluing, the curve $\hat{h}_{n,0}[1, r] \cup g'_n[r, s]$ is homotopic to the curve $\hat{h}_{n-1,0}[1, s]$ inside $D_s \setminus D_1$ relative the boundary points on circles. Therefore, the map obtained from gluing $\hat{h}_{n,0}$ and g'_n is homotopic to the map $\hat{h}_{n-1,0} : A(s) \to A(\tilde{s})$. If we denote the lift of g'_n via ϕ_1 and $\tilde{\phi}_1$ by g_n, this homotopy can be lifted to a homotopy between $h_{n-1,0}$ and the map obtained from gluing g_n and $h_{n,0}$.

Before we define k_n for the other cases, we need to show that the q.c. map $\mathbf{h}'_{n-1}$ built in Case $\mathscr{C}$ has an appropriate q.c. extension onto $\mathbf{L}_n$.

Lemma 4.13. *The q.c. map $\mathbf{h}'_{n-1}$ introduced in Case $\mathscr{C}$ has a q.c. extension onto $\mathbf{L}_n$ with a uniform bound on its dilatation depending only on ε. Moreover, this extension can be made homotopic to $\boldsymbol{\psi}_{n-1}$ relative the Julia bouquet $\mathbf{B}_{2,0}$.*

Proof. Consider the annuli $S_{n-1}(U_{n,0} \setminus V_{n,0})$ and $\widetilde{S}_{n-1}(\widetilde{U}_{n,0} \setminus \widetilde{V}_{n,0})$ for the first renormalizations of $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$. Let

$$g_n : S_{n-1}(U_{n,0} \setminus V_{n,0}) \to \widetilde{S}_{n-1}(\widetilde{U}_{n,0} \setminus \widetilde{V}_{n,0})$$

be a q.c. map which satisfies the equivariance relation on the boundaries of these annuli. By lifting g_n onto the preimages of these annuli we obtain a q.c. map g_n from complement of the little Julia set $J_{1,0} := J(\mathcal{R}^1(\mathbf{f}_{c_{n-1}}))$ to the complement of the little Julia set $\widetilde{J}_{1,0} := J(\mathcal{R}^1(\mathbf{f}_{\tilde{c}_{n-1}}))$ on the dynamic planes of $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$. By Lemma 4.5, g_n (or some rotation of it) can be extended as $\boldsymbol{\psi}_{n-1}$ onto $J_{1,0}$. Moreover these two maps are homotopic relative this little Julia set. We then adjust g_n so that it maps $\mathbf{L}'_n$ to $\widetilde{\mathbf{L}}'_n$.

Consider the three annuli $Y_0^0 \setminus \mathbf{L}_n$, $\mathbf{L}_n \setminus \mathbf{L}'_n$, and $\mathbf{L}'_n \setminus \mathbf{B}_{2,0}$, as well as the corresponding tilde ones. We have

$$\mathbf{h}_{n-1} : Y_0^0 \setminus \mathbf{L}_n \to \widetilde{Y}_0^0 \setminus \widetilde{\mathbf{L}}_n, \text{ and } \quad g_n : \mathbf{L}'_n \setminus \mathbf{B}_{2,0} \to \widetilde{\mathbf{L}}'_n \setminus \widetilde{\mathbf{B}}_{2,0}.$$

To glue these two maps on the middle annulus $\mathbf{L}_n \setminus \mathbf{L}'_n$, we use the above argument to find the right number of twists on this annulus. Consider a curve γ connecting a point a on the bouquet $\mathbf{B}_{2,0}$ to a point d on the boundary of Y_0^0 such that it intersects $\partial\mathbf{L}'_n$ and $\partial\mathbf{L}_n$ only once denoted by b and c, respectively. Lets denote by γ_{ab}, γ_{bc}, and γ_{cd} each segment of this curve cut off by these four points. The wrapping number $\omega(\boldsymbol{\psi}_{n-1}(\gamma))$ is uniformly bounded depending only on $\mathcal{SL}$ condition. That is because $\boldsymbol{\psi}_{n-1}$ depends continuously on $\tilde{c}_{n-1}$ and $\tilde{c}_{n-1}$ belongs to a compact class of parameters. Therefore, $\omega(\boldsymbol{\psi}_{n-1}(\gamma)) - \omega(\mathbf{h}_{n-1}(\gamma_{cd})) - \omega(g_n(\gamma_{ab}))$ is uniformly bounded, by Proposition 4.10, depending only on ε and the class $\mathcal{SL}$. If we glue $\mathbf{h}_{n-1}$ and g_n by such number of twists (see Lemma 4.11), the resulting map will be homotopic to $\boldsymbol{\psi}_{n-1}$ relative $\mathbf{B}_{2,0} \cup \partial Y_0^0$. □

As $\mathbf{h}'_{n-1}$ and $\boldsymbol{\psi}_{n-1}$ are identical on the boundary of Y_0^0, one can extend this map onto the other topological disks $\mathbf{L}_{n,i}$ as well. We denote this extended q.c. map by the same notation $\mathbf{h}'_{n-1}$.

If a Case $\mathscr{C}$ follows a Case $\mathscr{A}$ or $\mathscr{B}$, the number of twists k_n is defined as the one introduced in the above lemma. If level $n-1$ belongs

to Case $\mathscr{C}$ and level n is any of the three cases, we define k_n using uniformizations of the annuli $\mathbb{V}_{n,0} \setminus B_{n,0}$ and $\Delta_{n,0} \setminus B_{n,0}$, as well as the corresponding tilde ones instead of the above annuli.

The following elementary lemma is used in the final proof of isotopy. A proof of it is given in the Appendix.

Lemma 4.14. *Let U and $\widetilde{U}$ be closed annuli with outer boundaries γ_1 and $\tilde{\gamma}_1$ as well as inner boundaries γ_2 and $\tilde{\gamma}_2$, respectively. Also, let $h_1 : \gamma_1 \to \tilde{\gamma}_1$ be a homeomorphism and $h_2^t : \gamma_2 \to \tilde{\gamma}_2$, for $t \in [0,1]$, be a continuous family of homeomorphisms. Any one-to-one continuous interpolation $G^0 : U \to \widetilde{U}$ of h_1 and h_2^0 extends to a continuous family of one-to-one interpolations G^t of h_1 and h_2^t, for $t \in [0,1]$.*

Proposition 4.15. *The K-q.c. map H obtained from gluing all the maps $g_{n,i}$ and $h_{n,k}$ is homotopic to the topological conjugacy Ψ relative the postcritical set $\mathcal{PC}(f)$.*

Proof. Let H_n denote the q.c. map obtained from gluing all the maps $g_{1,0}$,$g_{2,k}, \dots, g_{n-1,l}$ and $h_{1,0}, h_{2,i}, \dots, h_{n,j}$, for all possible indices $k, \dots, l$, $i, \dots, j$.

First we claim that

- the maps H_1 and Ψ belong to the same homotopy class of maps from $\mathbb{C} \setminus \cup_i J_{1,i}$ to $\mathbb{C} \setminus \cup_i \widetilde{J}_{1,i}$, and
- the maps H_{n-1} and H_n, for every $n > 1$, belongs to the same homotopy class of maps from $\mathbb{C} \setminus \cup_i J_{n+1,i}$ to $\mathbb{C} \setminus \cup_i \widetilde{J}_{n+1,i}$.

By definition H_1 is $h_{1,0}$ which is homotopic to $\psi_{1,0}$, by Lemma 4.12 or Lemma 4.13. Therefore, it is homotopic to Ψ by Proposition 4.5.

Recall that the two maps H_{n-1} and H_n are identical on the complement of $\mathbb{V}_{n,j}$. Inside $\Delta_{n,0}$, H_{n-1} and H_n are $h_{n-1,0}$ and $h_{n,0}$, respectively.

The domain $\mathbb{V}_{n,0}$ is divided into an annulus $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$, and the topological disk $\Delta_{n,0}$. On $\Delta_{n,0}$, $h_{n,0}$ and $h_{n-1,0}$ are homotopic to $\psi_{n,0}$ relative $\cup_i J_{n+1,i}$ by Lemmas 4.12 and 4.13. Thus, there exists a homotopy h_n^t, for t in $[0,1]$, which starts with $h_{n,0}$ and ends with $h_{n-1,0}$, such that it maps $\partial \Delta_{n,0}$ to $\partial \widetilde{\Delta}_{n,0}$, for all $t \in [0,1]$. At time zero consider the map $h_{n,0}$ on the inner boundary of this annulus, $h_{n-1,0}$ on the outer boundary of this annulus and the interpolation $G_n^0 = g_{n,0}$ on the annulus. Applying above lemma with $h_{n-1,0}$ on the outer boundary and h_n^t on the inner boundary, we obtain a continuous family of interpolations G_n^t between them. The map G_n^1 is a homeomorphism from $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$ to $\widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{\Delta}_{n,0}$ which is an interpolation of $h_{n-1,0}$ on the boundaries. This interpolation must be homotopic to $h_{n-1,0}$ relative boundaries. That is because these two maps send a curve joining the two different boundaries to two curves (joining the two boundaries) which are homotopic

relative end points. This comes from our choice of the number of twists for the gluing maps.

Let $t_0 = 0 < t_1 < t_2 < \cdots$, be an increasing sequence in $[0,1]$ converging to 1. Assume that H^t, for t in $[t_0, t_1]$, denotes the homotopy obtained above between Ψ and H_1 relative the little Julia sets $J_{1,i}$. Also, let H^t, for $t \in [t_n, t_{n+1}]$, $n = 1, 2, \dots$, denote the homotopy between H_n and H_{n+1} relative the little Julia sets of level $n+2$.

It follows from the construction that $H^t(z)$ eventually stabilizes for any fixed z and equals to $H(z)$. Indeed, the *a priori* bounds assumption implies that the diameter of the topological disks $\mathbb{V}_{n,i}$ tends to zero as $n \to \infty$. Therefore, the uniform distance between H^t and H tends to zero as $t \to 1$. We conclude that H^t, for t in $[0,1]$, defines a homotopy between Ψ and H relative the postcritical set. Hence, H is a Thurston conjugacy between f and $\tilde{f}$. □

4.6. **Promotion to hybrid conjugacy.**

Proposition 4.16. *Suppose that all infinitely renormalizable unicritical polynomials in a given combinatorial class $\tau = \{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \dots\}$, satisfying $\mathcal{SL}$ condition, enjoy the a priori bounds. Then q.c. conjugacy implies hybrid conjugacy for maps in this class.*

Proof. Assume that there are polynomials P_1 and P_2 in such a combinatorial class which are q.c. equivalent but not hybrid equivalent. Define the set

$$\begin{aligned}\Omega :=& \{c \in \mathbb{C} : P_c \text{ is q.c. equivalent to } P_1\}\\ =& \{c \in \mathbb{C} : P_c \text{ is q.c. equivalent to } P_2\}.\end{aligned}$$

The plan is to show that the set Ω is both open and closed in $\mathbb{C}$ which is not possible.

Theorem 4.1 implies that q.c. conjugacy is equivalent to combinatorial conjugacy for maps in the class τ. Since every combinatorial class is an intersection of a nest of closed sets (connectedness locus copies), Ω is closed.

Consider a point P in Ω. The polynomial P is not hybrid equivalent to both of P_1 and P_2 by assumption. Assume that it is not hybrid equivalent to P_1 (for the other case just change P_1 to P_2). Let $\phi_1 : \mathbb{C} \to \mathbb{C}$, be a k-q.c. map with $\phi_1 \circ P = P_1 \circ \phi_1$. By pulling back the standard complex structure μ_0 on $\mathbb{C}$ under ϕ_1, we obtain a complex structure μ on $\mathbb{C}$ with dilatation bounded by $\frac{k-1}{k+1}$ and invariant under P. Consider the family of complex structures $\mu_\lambda := \lambda \cdot \mu$, for λ in the disk of radius $\frac{k+1}{k-1}$ at 0 in $\mathbb{C}$.

By the measurable Riemann mapping theorem [Ahl06], there are unique q.c. mappings $\phi_\lambda : \mathbb{C} \to \mathbb{C}$ with $\phi_\lambda^* \mu_\lambda = \mu_0$, $\phi_\lambda(0) = 0$, $\phi_\lambda(1) = 1$. The maps $P_\lambda := \phi_\lambda^{-1} \circ P_1 \circ \phi_\lambda$, for λ in the disk of radius $\frac{k+1}{k-1}$ at 0, preserve the standard complex structure μ_0. By Weyle's lemma ([Ahl06]), $P_\lambda : \mathbb{C} \to \mathbb{C}$ is a polynomial. As P_λ is conjugate to P, it is a degree d uniciritical polynomial and moreover, ϕ_λ maps the critical value of P to the critical value of P_λ. At $\lambda = 1$ we obtain the polynomial P, and at $\lambda = 0$ we obtain P_1 (both up to conformal conjugacy). By analytic dependence of the solution of the measurable Riemann mapping theorem on the complex structure, the conformal class of the family P_λ, for λ in the disk of radius $\frac{k+1}{k-1}$ at 0, covers a neighborhood of P in Ω. That is because, critical value of P_λ is equal to ϕ_λ(critical value of P), and ϕ_λ depends analytically on λ. This shows that P is an interior point in Ω, and therefore, Ω is open. □

5. Dynamical description of the combinatorics

In this section we give a detailed dynamical description of the combinatorial classes considered in the previous sections. Let c be an infinitely renormalizable parameter with sequence of renormalizations $f_n := \mathcal{R}^n(P_c)$, $n = 0, 1, 2, \ldots$, that are hybrid conjugate to $\mathbf{f}_{c_n}$. Let $Y_0^1(n)$ denote the critical puzzle piece of level 1 of $\mathbf{f}_{c_n}$. A dynamical meaning of a parameter c satisfying the *decoration* condition is that there exists a constant M such that for every $n \geq 0$ there are integers t_n and q_n, both bounded by M, with

- $\mathbf{f}_n^{kq_n}(0) \in Y_0^1(n)$, for every positive integer $k < t_n$,
- $\mathbf{f}_n^{t_n q_n}(0) \notin Y_0^1(n)$.

In particular, this condition implies that the number of rays landing at the dividing fixed point of $\mathbf{f}_{c_n}$ (here q_n) is uniformly bounded.

An infinitely renormalizable parameter is of bounded type if the relative return times t_{n+1}/t_n of the renormalizations $\mathcal{R}^n(f) = f^{t_n}$ are uniformly bounded by some constant M. It follows from definition that the decoration condition includes infinitely primitively renormalizable parameters of bounded type.

In Section 3.4, we associated a sequence of maximal connectedness locus copies $\tau(f) = \langle \mathcal{M}^1, \mathcal{M}^2, \ldots \rangle$ to every infinitely renormalizable unicritical polynomial-like map f. Let $\pi_n(\tau(f)) := \mathcal{M}^n$. Define

$$\tau(f, n) := \left\{ c \in \mathcal{M}_d \;\middle|\; \begin{array}{l} P_c(z) = z^d + c \text{ is at least } n \text{ times} \\ \text{renormalizable, and} \\ \pi_i(\tau(f)) = \pi_i(\tau(P_c)), \text{ for } i = 1, 2, \ldots, n \end{array} \right\}.$$

Given an infinitely renormalizable map f and a sequence of integers $n_0 = 0 < n_1 < n_2 < \cdots$, we define a sequence of relative connectedness locus copies of $\mathcal{M}_d$ as follows:

$$(\tilde{\tau}(f), \langle n_i \rangle) := \langle \tilde{\mathcal{M}}^{n_1}, \tilde{\mathcal{M}}^{n_2}, \dots, \tilde{\mathcal{M}}^{n_k}, \dots \rangle,$$

where,

$$\tilde{\mathcal{M}}^{n_k} := \tau(\mathcal{R}^{n_{k-1}} f, n_k - n_{k-1}).$$

Given a sequence of integers $n_0 = 0 < n_1 < n_2 < \cdots$, one can see that there is a one to one correspondence between the two sequences $\tau(f)$ and $(\tilde{\tau}(f), \langle n_i \rangle)$. Thus, one may take the latter one as definition of the combinatorics of an infinitely renormalizable map.

Consider the main hyperbolic component of the connectedness locus $\mathcal{M}_d$. There are infinitely many, primary, hyperbolic components of $\mathcal{M}_d$ attached to this component (corresponding to rational numbers). Similarly, there are infinitely many hyperbolic components, secondary ones, attached to these primary components, and so on. Consider the set of all hyperbolic components obtained this way, i.e., the ones that can be connected to the main hyperbolic component by a chain of hyperbolic components bifurcating one from another. The closure of this set plus all possible components of its complement is called the *molecule* $\mathscr{M}_d$.

We say that an infinitely renormalizable map f satisfies the molecule condition, if there exists a positive constant $\eta > 0$ and an increasing sequence of positive integers $n_0 = 0 < n_1 < n_2 < \cdots$, such that

- for every $i \geq 1$, $\mathcal{R}^{n_i} f$ is a primitive renormalization of $\mathcal{R}^{n_i - 1} f$, and
- the Euclidean distance between $\tilde{\mathcal{M}}_d^{n_i}$ and the molecule $\mathscr{M}_d$ is at least η.

Note that for a map satisfying this condition, there may be infinitely many satellite renormalizable maps in the sequence $\langle \mathcal{R}^n f \rangle$. However, the condition requires that there are infinite number of primitive levels with the corresponding relative connectedness locus copies uniformly away from the molecule. By a compactness argument, one can see that the parameters in the decoration condition satisfy the molecule condition.

For every $\varepsilon \geq 0$, and every hyperbolic component of the connectedness locus, there are at most finite number of limbs attached to this hyperbolic component with diameter bigger than ε (by Yoccoz inequality on the size of the limbs [Hub93]). This implies that for every $\eta > 0$, all the secondary limbs except finite number of them are contained in

the η neighborhood of the molecule. This implies that the parameters satisfying the molecule condition also satisfy the $\mathcal{SL}$ condition. Therefore, combining with [KL07] and [KL08] we obtain the corollary stated in the Introduction.

Appendix A.

Proof of Proposition 4.5. Consider an external ray R landing at a non-dividing fixed point β_0 of f. As this ray is invariant under f, and ϕ commutes with f, we have $f \circ \phi(R) = \phi(R)$. Thus, $\phi(R)$ is also invariant under f which implies that $\phi(R)$ lands at a non-dividing fixed point β_j of f. Now, there exists ρ_j such that $\rho_j(\phi(R))$ lands at β_0. Let ψ denote the map $\rho_j \circ \phi$, and R' denote the ray $\rho_j(\phi(R))$. For such a rotation ρ_j, ψ also commutes with f, and R' is also invariant under f.

The external ray R cuts the annulus $V_1 \setminus V_2$ into a quadrilateral $I_{0,1}$. The preimage $f^{-1}(I_{0,1})$ produces d quadrilaterals denoted by

$$I_{1,1}, I_{1,2}, \ldots, I_{1,d},$$

ordered clockwise starting with R. Similarly, the f^n-preimage of $I_{0,1}$ produces d^n quadrilaterals $I_{n,1}, I_{n,2}, \ldots, I_{n,d^n}$ (also ordered clockwise starting with R). In the same way, the external ray R' produces quadrilaterals denoted by $I'_{n,j}$, ordered clockwise starting with R'. First we show that the Euclidean diameter of $I_{n,j}$ (and $I'_{n,j}$) goes to zero as n tends to infinity.

Denote $f^{-i}(V_1)$ by V_{i+1}, and let d_{i+1} denote the hyperbolic metric on the annulus $V_{i+1} \setminus K(f)$. As $I_{n,j} \subseteq V_n$, and the intersection of the nest of topological disks V_n is equal to $K(f)$, the quadrilaterals $I_{n,j}$ converge to the boundary of $V_1 \setminus K(f)$ as n goes to infinity. In order to show that the Euclidean diameter of these quadrilaterals go to zero, it is enough to show that their hyperbolic diameters in $(V_1 \setminus K(f), d_1)$ stay bounded. Since $f^{n-1} : (V_n \setminus K(f), d_n) \to (V_1 \setminus K(f), d_1)$ is an unbranched covering of degree d^{n-1}, it is a local isometry. As closure of $f^{n-1}(I_{n,j})$ is a compact subset of $V_1 \setminus K(f)$, we conclude that $I_{n,j}$ has bounded hyperbolic diameter in $(V_n \setminus K(f), d_n)$. Finally, contraction of the inclusion map from $(V_n \setminus K(f), d_n)$ into $(V_1 \setminus K(f), d_1)$ implies that $I_{n,j}$ has bounded hyperbolic diameter in $(V_1 \setminus K(f), d_1)$.

With a similar argument, one can show that the hyperbolic distance between $I_{n,j}$ and $I'_{n,j}$ inside $(V_1 \setminus K(f), d_1)$ is also uniformly bounded.

Since ψ is a conjugacy, it sends $I_{n,j}$ to $I'_{n,j}$. Therefore, as w converges to $K(f)$, w and $\psi(w)$ belong to $I_{n,j}$ and $I'_{n,j}$, respectively, with larger and larger n. Combining with the above argument, we conclude that the Euclidean distance between these two points tends to zero. This

implies that ψ can be extended as the identity map on the filled Julia set. $\square$

Proof of Lemma 4.11. Let $\Pi_r := \{z \mid 0 \leq \mathrm{Im}(z) \leq \frac{1}{2\pi}\log r\}$, for $r > 1$, denote the covering space of $A(r)$ with the deck transformation group generated by $z \to z+1$. Similarly $\Pi_{r'}$ denotes the covering space of $A(r')$ with the same deck transformation group. We may assume that $\log r$ and $\log r'$ are at least 6π. Otherwise, one may rescale these strips under affine maps of the form $(x, y) \mapsto (x, ay)$, for some real constant a, which have uniformly bounded dilatation, and continue with the following argument.

The homeomorphisms h_1 and h_2 lift to 1-periodic homeomorphisms

$$\hat{h}_1 : \{z \mid \mathrm{Im}(z) = \frac{1}{2\pi}\log r\} \to \{z \mid \mathrm{Im}(z) = \frac{1}{2\pi}\log r'\}, \text{ and}$$
$$\hat{h}_2 : \{z \mid \mathrm{Im}(z) = 0\} \to \{z \mid \mathrm{Im}(z) = 0\},$$

with $\hat{h}_2(0) = 0$ and $\hat{h}_1(\frac{\mathbf{i}}{2\pi}\log r) = \theta(h_1(r)) - \theta(h_2(1)) + 2\pi k + \frac{\mathbf{i}}{2\pi}\log r'$. As these maps have K_2-q.c. extension to some neighborhood of their domains, they are quasi-symmetric with a constant $M(K_2)$ depending only on K_2 (See Theorem 1 in [Ahl06], Page 40).

To prove the lemma, it is enough to introduce a 1-periodic q.c. mapping $h : \Pi_r \to \Pi_{r'}$ matching $\hat{h}_1$ and $\hat{h}_2$ on the boundaries and with a uniform bound on its dilatation in terms of the parameters.

Define the map $\phi : \Pi_\infty \to \Pi_\infty$ as $\phi(x, y) := u(x, y) + \mathbf{i}v(x, y)$, where

$$u(x, y) := \frac{1}{2y}\int_{-y}^{+y} \hat{h}_2(x+t)\, dt,$$
$$v(x, y) := \frac{1}{2y}\int_{0}^{y} (\hat{h}_2(x+t) - \hat{h}_2(x-t))\, dt.$$

It has been proved in [Ahl06], Page 42, that ϕ is a q.c. mapping with dilatation depending on $M(K_2)$. Note that ϕ is 1-periodic in the first variable. It follows from Lemma 3 on Page 41 of [Ahl06] that

$$v(0, 1) \leq \int_0^1 \hat{h}_2(t)\, dt - \frac{1}{2} \leq \frac{M(K_2)}{M(K_2)+1} \leq \frac{1}{2} \in [0, 3].$$

Also,

$$v_x(x, 1) = 0, \text{ and } u_x(x, 1) = 1, \text{ for } -\infty < x < \infty.$$

This implies that ϕ maps the horizontal line through $\mathbf{i}$ to a horizontal line in $\Pi_{r'}$ as a translation. Similarly one extends $\hat{h}_1$ to a q.c. mapping ψ of the set $\{z \mid \mathrm{Im}(z) \leq \frac{1}{2\pi}\log r\}$ such that it maps the horizontal line through $\frac{\mathbf{i}}{2\pi}\log r - 1$ to a horizontal line in $\Pi_{r'}$ as a translation.

Consider the map $H : \{z \mid 1 \le \mathrm{Im}(z) \le \frac{1}{2\pi}\log r - 1\} \to \Pi_{r'}$ defined as

$$(x, y) \mapsto (1-y)\phi(x, 1) + y\psi(x, \frac{1}{2\pi}\log r - 1).$$

The homeomorphism H is an affine map with dilatation depending only on k, and ϕ as well as ψ are q.c. mappings with dilatations depending only on $M(K_2)$. It follows from analytic definition of q.c. mappings that the homeomorphism

$$\begin{cases} \phi(x,y) & \text{if } 0 \le x \le 1 \\ H(x,y) & \text{if } 1 \le x \le \frac{1}{2\pi}\log r - 1 \\ \psi(x,y) & \text{if } \frac{1}{2\pi}\log r - 1 \le x \le \frac{1}{2\pi}\log r \end{cases}$$

is q.c. with dilatation depending only on k and $M(K_2)$. □

Proof of Lemma 4.14. By lifting h_1, h_2^t, and G^0 to the covering space $\Pi := \mathbb{R}\times[0,1]$ of U and $\widetilde{U}$, we obtain homeomorphisms $\hat{h}_1 : \mathbb{R}\times\{0\} \to \mathbb{R}\times\{0\}$, $\hat{h}_2^t : \mathbb{R}\times\{1\} \to \mathbb{R}\times\{1\}$, and $\hat{G}^0 : \Pi \to \Pi$, 1-periodic in the first coordinate. By adding a constant, we may assume that $\hat{h}_1(0,0) = (0,0)$. This also uniquely determines $\hat{G}^0$ as an extension of $\hat{h}_1$ and then $\hat{h}_2$. Define the continuous family of 1-periodic homeomorphisms $T^t : \Pi \to \Pi$, for $t \in [0,1]$, as follows:

$$T^t(x,y) := (1-y)\cdot\hat{h}_1(x,0) + y\cdot\hat{h}_2^t(x,1).$$

If we define the 1-periodic homeomorphism $H : \Pi \to \Pi$ as $(\hat{G}^0)^{-1} \circ T^0$, then one can verify that $T^t \circ H^{-1}$ is a homotopy between $\hat{h}_1$ and $\hat{h}_2^t$ starting with $\hat{G}^0$. One projects this periodic family to obtain a continuous family of interpolations on U with the desired properties. □

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